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THE
ELEMENTARY PRINCIPLES
OF
MECHANICS.

VOL. III.
KINETICS.

BY

A. JAY DU BOIS, C.E., PH.D.,
Professor of Civil Engineering
in the Sheffield Scientific School of Yale University;
Author of "Elements of Graphical Statics,"
"The Strains in Framed Structures,"
etc.

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PREFACE.

THE plan and method of presentation of this work are the result of many years of teaching experience. It has worked well in the hands of the author and has grown into its present form gradually, in response to the needs of student and teacher. It would seem therefore not unreasonable to hope that it may be of service to others in similar circumstances and with similar needs.

It is the confirmed opinion of the writer that a text-book to be of the highest value to the student should contain more than a brief outline of the most elementary principles, even although the time at disposal for actual class work may oblige the teacher to confine himself to such an outline. A text-book should be of use to the student during the whole of his course and afterwards as a book of reference, as the various applications of mechanics oblige him to look up more in detail the underlying principles. With such a text-book in his possession the student grows in the knowledge and mastery of the subject long after his class has passed on, and such a work becomes a valuable possession. His growing familiarity with its scope and uniform notation make it easy of reference, and it forms the best preparation for reading with ease and intelligence more advanced works. He should find in it everything in the way of principle he may need, with enough of practical application to illustrate and guide him.

For the teacher the work should be so arranged that he can readily lay out his course according to the time at his disposal and the needs of his students.

In most of our technical institutions the calculus is taken up at an early stage. It is very desirable that its applications should be kept in view. If, then, the study of mechanics can be taken up earlier than is customary, the two subjects will illustrate each other to mutual advantage.

The present work is an attempt to cover the points indicated.

The subject of Kinematics is treated in Vol. I with all the thoroughness its importance demands, and intentionally with more fulness than the time ordinarily at disposal for this subject would warrant. We would, however, call attention to the following points: Articles for advanced students are in small type. Articles containing applications of the calculus are enclosed in brackets. The large type by itself constitutes an abridged course. The teacher can thus lay out such a course as regards time and thoroughness as he desires. As a work of reference for both student and teacher it should have an independent value. It is believed that the engineering student will seldom need to go outside for reference.

The examples are numerous. The value to the student of these is, in the writer's opinion, very great. They are kept apart from the text, and the teacher can make use of them to such extent as seems good to him. Those who may be at first sight deterred from using the book, because of its covering more ground than the time at their disposal warrants, are asked to look it over in the light of the preceding explanation. It is believed that no matter how abridged a course is thought necessary, the teacher will find it here ready to his hand.

The subject of Statics is treated in Vol. II, and here the same remarks hold. The discriminating teacher who wishes a short practical course will easily find it here. A large portion of the volume is devoted to engineering applications, such as Dams, Earthwork, Retaining Walls, Strength and Elasticity of Materials, Theory of Flexure. All these subjects are taught with more or less fulness, usually by the use of separate works. The subjects named are treated with thoroughness and fully illustrated by numerous examples, within the compass of 160 pages.

The subject of Kinetics is treated in the third volume according to the same plan.

The volumes have been printed separately for convenience of class work. Each is furnished with a complete index for reference. We have included in the three volumes no more of the subject than in our opinion the engineering student should sooner or later be familiar with. "If a man knows mechanics," says a well-known engineer, "he can make himself an engineer; if he does not, nobody can make an engineer of him."

There are two ways of teaching this subject. One by the use of an elementary abridgment, which student and teacher must afterward supplement by the use of more advanced works later on. The student then throws aside his elementary treatise as soon as he finishes it, and either has to start again with a new work, or, as is often the case, gets no real grasp of the subject and no working familiarity with any valuable reference-work.

The other way is that here pursued, of giving the student a work sufficient for his needs, written with direct reference to his needs and development from beginning to end, while at the same time it allows of all desirable curtailment in the introductory work. It is so presented that the subject may be taken up earlier than is customary and made to illustrate and help in the other mathematical studies of the course.

It is the author's plan, with his students, to first go over all three volumes, omitting everything not absolutely essential, taking thus only selected portions of the large print. When the student thus has a connected and intelligent grasp of the whole subject, the remaining time is employed in picking up those omitted portions which are of most importance. The work is then in the student's hands as a work of reference which he can use, and which he is safe to use and value during the rest of his course.

Heretofore, works thus valuable for reference have been too condensed and too abstract. They are written for the engineer, not for the student. On the other hand, text-books for the student have been too elementary and too much abridged. They are intended for the beginner only and have no value for the progressive student. The present work is an attempt to take the beginner and leave him with a work which shall be of permanent value to him during and after his entire course.

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SECTION I. KINETICS OF A PARTICLE.

CHAPTER I.

EQUATION OF FORCE.

KINETICS. TRANSLATION. EQUATION OF FORCE. EQUATIONS OF MOTION.

Kinetics.—We have treated in Vol. I of the science of **Kinematics** (κινημα, motion), or the measurable relations of space and time only, that is, of pure motion. We have therefore considered the motion of a point or of a system of points without reference to matter or force. But we have to deal in nature with force and material bodies.

The science which treats of those measurable relations of matter, space, and time involved in the study of the change of motion of bodies due to force is called **Dynamics** (δύναμις, force).

That portion of Dynamics which treats of those principles which are necessary for the discussion of forces and bodies in equilibrium, and generally of forces without reference to the change of motion caused by them, we have called **Statics**, and have considered in Vol. II.

That portion of Dynamics which treats of forces with reference to the change of motion caused by them we call **Kinetics**, and treat in the present volume.

Kinetics of a Particle—Translation.—We have seen (page 83, Vol. II, *Statics*) that when a rigid body is acted upon by any number of forces applied at different points and acting in different directions, that is, whatever the motion of the body may be, *the motion of the centre of mass is precisely the same as if the body were replaced by a particle of equal mass at the centre of mass, and all the forces were transferred to this particle without change of direction or magnitude.*

When, therefore, we consider only the motion of translation of a body without reference to its rotation, if any, *we can always consider the body as a particle of equal mass concentrated at the centre of mass.*

The study of motion of translation of a body under the action of force is therefore the same as that of a particle of equal mass. We thus consider first the **Kinetics of a Particle**, or **Translation.**

Equation of Force.—The student before taking up this portion of our work should be familiar with the general principles of Dynamics as given in Vol. II, *Introduction*, Chapters I to IV, pages 1 to 55.*

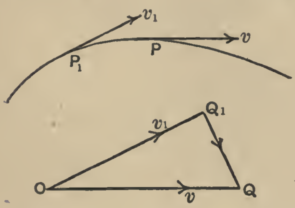
We have there seen how to measure mass and force, and how to find the centre of mass of a body. We have also seen that the direction of a force is the same as that of the *acceleration* it causes, and the magnitude of the force is proportional to the acceleration it causes. The relation between force, mass and acceleration is there found to be given by

$$F = mf, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

where m is the mass of a particle in units of mass and f is the instantaneous acceleration in units of acceleration and F is the force in units of force.

This is called the equation of force. It gives the magnitude of the force F which causes in any particle of mass m the acceleration f in the direction of the force. Force, then, has magnitude and direction and can be represented like acceleration by a straight line. The principles, then, of pages 70, 84, 95, Vol. I, *Kinematics*, hold good for forces also, and we can resolve and combine forces and have the "triangle and polygon of forces."

Thus let v_1 be the initial velocity of a particle P of mass m, moving in any path P_1P, and v its final velocity at the end of any time t.



If we draw OQ_1 parallel and equal to v_1 and OQ parallel and equal to v, then Q_1Q gives the *integral* or *entire acceleration* in the time t, both in direction and magnitude, and $\frac{Q_1Q}{t}$ gives the magnitude and direction of the *mean* or *average acceleration*, or mean time-rate of change of velocity in the time t.

The limiting magnitude and direction of $\frac{Q_1Q}{t}$ when the time t is indefinitely small, is the acceleration f, or instantaneous time-rate of change of velocity at any instant.

Now this change of velocity is due to the force. If there were no force, v_1 would remain unchanged *both in magnitude and direction.* The force at any instant is then given in magnitude by

$$F = mf,$$

and its direction is the direction of f.

Force, then, like acceleration, is *uniform* when it does not change either in magnitude or direction. If either magnitude or acceleration changes it is *variable.*

The unit of force is then that force which gives one unit of mass one unit of acceleration in the direction of the acceleration. This is called the *poundal* when the unit of acceleration is 1 ft.-per-sec. per sec. and the unit of mass is the pound. If then in equation (I) we take m in pounds and f in feet-per-second per second, equation (I) *always gives F in poundals.*

---

* Any student taking this work in course should not fail to thoroughly review these pages at this point before going farther.

Since the acceleration due to gravity is g ft.-per-sec. per sec., the force of gravity upon one pound *is then g poundals.* One pound then weighs g poundals. Hence if we wish F in gravitation units (page 6, Vol. II, *Statics*), since one pound weighs g poundals we have only to divide mf in equation (I) by g. We have then

$$F = \frac{mf}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(II)}$$

In equation (II), if we take m in pounds and f in feet-per-second per second, we have *F in gravitation units;* that is, we obtain *the number of pounds whose weight will be equal to the force.*

If in equation (I) we take m in grams and f in centimetres-per-second per second, *we always obtain F in dynes;* while equation (II) gives us F in gravitation units, that is, *the number of grams whose weight will be equal to the force.*

The accurate value of g for the locality should be taken when great accuracy is required. The values of g for different localities are given on page 93, Vol. I, *Kinematics.* In ordinary calculations g is usually taken at $32\frac{1}{6}$ ft.-per-sec. per sec., or 981 cm.-per-sec. per sec. We see then that one poundal is the weight of about one half an ounce, or more strictly the weight of $\frac{1}{g}$ part of a pound, while one dyne is the weight of about 1 milligram, or more strictly the weight of $\frac{1}{g}$ part of a gram, g in the first case being taken in feet-per-second per second, and in the second case in centimetres-per-second per second.

Equation of Motion.—The equation of force $F = mf$ enables us to find in any case any one of the three quantities, acting force F, mass of particle m, or acceleration f, if the other two are given.

If then mass and force are given, the acceleration $f = \frac{F}{m}$ can be found. We can then apply the equations of motion given in *Kinematics*, Vol. I, Chap. VII, page 81.

Illustrations.—The following illustrations will make clear the application of the preceding article.

1. Motion of a Particle Projected Vertically Upwards from the Surface of the Earth.—Take the origin at the starting-point on the earth's surface. Let the mass of the particle be m and the acceleration of gravity at the earth's surface be g. Then at or near the earth's surface the weight of the particle is $F = -mg$, where the minus sign denotes that the force acts towards the origin (page 58, Vol. II, *Statics*).

We have then for the acceleration $f = \frac{F}{m}$ or

$$f = -g.$$

If the initial velocity is v_1, we have then for the velocity v at the end of any time t

$$v = v_1 - gt.$$

We then proceed as on page 93, Vol. I, *Kinematics.*

[In calculus notation we have

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -g.$$

We then proceed as on page 94, Vol. I, *Kinematics.*]

2. Motion of a Particle Projected Vertically Downwards towards the Surface of the Earth.—Take the origin at the starting-point. Let the mass of the particle be m and the acceleration of gravity at the earth's surface be g. Then at or near the earth's surface the weight of the particle is $F = +mg$, where the plus sign denotes that the force acts away from the origin (page 58, Vol. II, *Statics*).

We have then for the acceleration $f = \frac{F}{m}$ or

$$f = +g.$$

If the initial velocity is v_1, we have then for the velocity v at the end of any time t

$$v = v_1 + gt.$$

We then proceed as on page 93, Vol. I, *Kinematics.*

[In calculus notation we have

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = +g.$$

We then proceed as on page 94, Vol. I, *Kinematics.*]

3. Motion of a Falling Particle at a Great Distance from the Earth's Surface—Resistance of Air neglected.—Take the origin at the earth's surface, and let m as before be the mass of the particle and the acceleration of gravity at the earth's surface be g. Then at the earth's surface the weight of the particle is $-mg$, where the minus sign denotes direction towards the origin (page 58, Vol. II, *Statics*). Since, by Newton's law (page 44, Vol. II, *Statics*), the force of gravitation is inversely as the square of the distance, we have for the force F at any distance s, if r' is the earth's radius,

$$F : -mg :: r'^2 : s^2, \quad \text{or} \quad F = -\frac{mgr'^2}{s^2},$$

where the minus sign denotes that the force acts towards the origin.

We have then for the acceleration $f = \frac{F}{m}$ or

$$f = -\frac{gr'^2}{s^2}.$$

We then proceed as on page 99, Vol. I, *Kinematics.*

[In calculus notation

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -\frac{gr'^2}{s^2}.$$

We then proceed as on page 100, Vol. I, *Kinematics.*]

4. Motion of a Particle Falling under the Action of Gravity near the Earth's Surface in a Resisting Medium.—Take the origin at the starting-point. Let V be the volume of the particle and δ its density, and $\varDelta$ the density of the medium. Then the mass of the particle is $m = \delta V$ and its weight is $+ mg = + \delta Vg$, where g is the acceleration of gravity at the earth's surface and the plus sign denotes direction away from the origin (page 58, Vol. II, *Statics*).

The buoyant force of the medium is, according to a well-known principle of Physics, equal to the weight of an equal volume of the medium, or $- \varDelta Vg$, the minus sign denoting direction towards the origin. The resultant force is then

$$F = + \delta Vg - \varDelta Vg = \delta Vg\left(1 - \frac{\varDelta}{\delta}\right).$$

We have then for the acceleration, apart from the resistance of the medium, $f = \frac{F}{m}$ or

$$f = g\left(1 - \frac{\varDelta}{\delta}\right).$$

[In calculus notation

$$f = \frac{dv}{dt} = \frac{d^2s}{dt_2} = g\left(1 - \frac{\varDelta}{\delta}\right).$$

We then proceed as on page 111, Vol. I, *Kinematics*.]

5. Motion of a Particle Projected Upwards under the Action of Gravity near the Earth's Surface in a Resisting Medium.—Take the origin at the starting-point, which in this case is the earth's surface. The mass of the particle as before is $m = \delta V$, and its weight is $- mg = - \delta Vg$, where the minus sign denotes direction towards the origin (page 58, Vol. II, *Statics*).

The buoyant force is as before $+ \varDelta Vg$, where the plus sign denotes direction away from the origin. The resultant force is then in this case

$$F = - \delta Vg + \varDelta Vg = - \delta Vg\left(1 - \frac{\varDelta}{\delta}\right).$$

We have then for the acceleration, apart from the resistance of the medium, $f = \frac{F}{m}$ or

$$f = - g\left(1 - \frac{\varDelta}{\delta}\right).$$

[In calculus notation

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = - g\left(1 - \frac{\varDelta}{\delta}\right).$$

We then proceed as on page 112, Vol. I, *Kinematics*.]

6. Motion of a Particle in a Straight Line under the Action of an Attractive Force proportional to the Distance of the Particle from a Fixed Point in the Line of Motion.—Take the fixed point as origin. Let m be the mass of the particle and a' its acceleration at the distance r'. Then the force at the distance r' is $- ma'$, where the minus sign denotes direction towards the origin (page 58, Vol. II, *Statics*). Hence we have for the force at any other point at a distance s

$$F : - ma' :: s : r', \quad \text{or} \quad F = - \frac{ma'}{r'}s.$$

We have then for the acceleration $f = \frac{F}{m}$ or

$$f = -\frac{a'}{r'}s.$$

The motion is therefore *simple harmonic* (page 101, Vol. I, *Kinematics*).

[In calculus notation

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -\frac{a'}{r'}s.$$

We then proceed as on page 106, Vol. I, *Kinematics.*]

7. Motion of a Particle in a Straight Line under the Action of a Repulsive Force proportional to the Distance of the Particle from a Fixed Point in the Line of Motion.—Take the fixed point as origin, let m be the mass of the particle and a' its acceleration at the distance r'. Then the force at the distance r' is $+ ma'$, where the plus sign denotes direction away from the origin (page 58, Vol. II, *Statics*). Hence we have for the force at any other point at a distance s

$$F : ma' :: s : r' \quad \text{or} \quad F = +\frac{ma'}{r'}s.$$

We have then for the acceleration $f = \frac{F}{m}$ or

$$f = +\frac{a'}{r'}s.$$

The motion is therefore simple harmonic (page 101, Vol. I, *Kinematics*).

[In calculus notation

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = +\frac{a'}{r'}s.$$

We then proceed as on page 107, Vol. I, *Kinematics.*]

The preceding illustrations, together with the following examples, will make clear the application of the equation of force

$$F = mf$$

to cases of motion of translation or motion of material particles.

EXAMPLES.

(1) *A spring-balance is graduated for a place where $g = 32.2$ ft.-per-sec. per sec. and indicates* 1.6 *pounds at a place where $g = 32$ ft.-per-sec. per sec. Find the correct value of the mass.*

Ans. If m is the actual mass, $m \times 32$ is the actual weight in poundals of that mass at the place where it is weighed.

If the balance is correctly graduated, its indicated mass × 32.2 ought to give the actual weight. Hence $m \times 32 = 1.6 \times 32.2$, or $m = 1.61$ lbs.

(2) *A uniform force of* 2 *lbs. acts on a particle of* 40 *lbs. mass for half a minute. Find the velocity acquired and the space passed through.* ($g = 32$.)

Ans. Since the force is uniform, f is constant in direction and magnitude. The force is the weight of 2 lbs. or $2g$ poundals. By equation of force $2g = 40f$, or $f = \frac{g}{20} = 1.6$ ft.-per-sec. per sec. Since f is uniform, equations page 51, Vol. I, *Kinematics*, apply, and we have

$$v = ft = 48 \text{ ft. per sec.}, \quad s = \frac{1}{2}ft^2 = 720 \text{ ft.}$$

(3) *A particle acted upon by a uniform force describes in ten seconds, starting from rest, a distance of* 25 *ft. Compare the force with the weight of the body and find the velocity acquired.* $(g = 32.)$

Ans. $s = \frac{1}{2}ft^2$ (page 51, Vol. I, *Kinematics*). Hence $f = \frac{2s}{t^2} = \frac{50}{100} = 0.5$ ft.-per-sec. per sec.; $v = ft = 5$ ft. per sec.; $F = mf$, or $\frac{F}{mg} = \frac{f}{g} = \frac{0.5}{32} = \frac{1}{64}$.

(4) *In what time will a force which is equal to the weight of a pound move a mass of* 18 *pounds through* 50 *ft. along a smooth horizontal plane, and what will be the velocity acquired?* $(g = 32.)$

Ans. (Page 51, Vol. I, *Kinematics*.) $v = ft$, $s = \frac{1}{2}ft^2$, $F = mf$. Hence

$$f = \frac{g}{18} = \frac{16}{9} \text{ ft.-per-sec. per sec.}; \quad t = 7\tfrac{1}{2} \text{ sec.}; \quad 13\tfrac{1}{3} \text{ ft. per sec.}$$

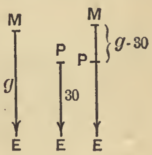
(5) *A mass of* 224 *lbs. is placed on a smooth horizontal plane and a uniform force acting on it parallel to the plane for* 5 *sec. causes it to describe* 50 *ft. in that time. Show that the force is equal to about* 28 *lbs. weight.* $(g = 32.)$

(6) *Forces of* 20 *and* 30 *units acting on two particles produce accelerations of* 40 *and* 50 *units respectively. Show that the masses are as* 10 *to* 12.

(7) *Two forces produce in two particles accelerations of* 25 *and* 30 *units respectively. Show that if the masses are equal the forces are as* 5 *to* 6, *and that if the forces are equal the masses are as* 6 *to* 5.

(8) *A mass of* 20 *lbs. is placed upon a horizontal plane which is made to descend with a uniform acceleration of* 30 *ft.-per-sec. per sec. Find the pressure on the plane due to the mass.* $(g = 32.2.)$

Ans. Acceleration of the mass with reference to the earth is g; of the plane relative to the earth 30. Acceleration of the mass relative to the plane is $g - 30 = 2.2$ ft.-per-sec. per sec. Pressure $= 2.2 \times 20 = 44$ poundals or weight of $\frac{44}{32.2} = 1.36$ lbs.



(9) *A mass of* 20 *lbs. rests on a horizontal plane which is made to ascend, first, with a constant velocity of* 1 *ft. per second; second, with a velocity constantly increasing at the rate of* 1 *ft.-per-sec. per sec. Find in each case the pressure on the plane.* $(g = 32.)$

Ans. In the first case the pressure is the weight of 20 lbs. simply. In the second case the acceleration of the mass relative to the plane is $g + 1 = 33$ ft.-per-sec. per sec. Pressure $= 20 \times 33 = 660$ poundals or weight of $\frac{660}{32} = 20\frac{10}{16}$ lbs.

(10) *A balloon is ascending vertically with a velocity which is increasing at the rate of* 3 *ft.-per-sec. per sec. Find the apparent weight of* 1 *lb. weighed in the balloon by means of a spring-balance.* $(g = 32.2.)$

Ans. 1.093 lbs.

(11) *A mass M lies on a smooth horizontal plane. A uniform horizontal force F is continuously applied. How long will it take to move the mass s ft. from rest? Take $M = 2240$ lbs., $F = 28$ lbs., $s = 5$ ft.* $(g = 32.)$

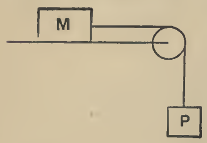
Ans. $F = Mf$ poundals; hence $f = \frac{F}{M}$ or $f = \frac{28g}{2240} = \frac{1}{80}g$ ft.-per-sec. per sec.

If $g = 32$ ft.-per-sec. per sec., we have $f = \frac{2}{5}$ ft.-per-sec. per sec.

Since f is uniform, we have (page 51, Vol. I, *Kinematics*)

$$s = \frac{1}{2}ft^2, \quad \text{or} \quad 5 = \frac{1}{2} \times \frac{2}{5}t^2, \quad \text{or} \quad t = 5 \text{ sec.}$$

(12) *Let the mass $M = 2240$ lbs. be moved by a rope which passes over the edge of the plane on a pulley and sustains a mass $P = 28$ lbs. at its other end. Disregarding all friction and mass of pulley and rope and supposing the rope perfectly flexible and inextensible, find how long it will take to move the mass M a distance $s = 5$ ft. from rest.* $(g = 32.)$



Ans. The student should carefully compare this example with the preceding and following.

Here the tension on the rope is $F = Mf$ poundals, where f is the acceleration of M. Since P has the same acceleration downward, the resultant acceleration of P is $g - f$. Hence the tension on the rope is also $P(g - f)$ poundals. Therefore

$$Mf = P(g - f), \quad \text{or} \quad f = \frac{Pg}{P + M} = \frac{28g}{2268} = \frac{32}{81} \text{ ft.-per-sec. per sec.}$$

Or we may obtain the same result as follows: The *moving force* is the weight of P or the attraction of gravity for P, or Pg poundals, or the weight of 28 lbs. as in Ex. 11. The mass moved is $P + M$. Hence

$$(P + M)f = Pg, \quad \text{or} \quad f = \frac{Pg}{P + M}.$$

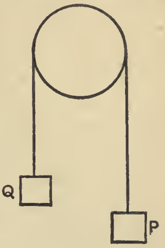
We have for uniform acceleration (page 51, Vol. I, *Kinematics*)

$$s = \frac{1}{2}ft^2, \quad \text{or} \quad 5 = \frac{1}{2} \times \frac{32}{81}t^2, \quad \text{or} \quad t = 5.031 \text{ sec.}$$

The tension on the rope is Mf or $P(g - f)$ or $\frac{PMg}{P + M} = \frac{2240 \times 32}{81}$ poundals, or the weight of $\frac{PM}{P + M} = \frac{2240}{81} = 27\frac{53}{81}$ lbs.

(13) *Two masses $P = 2240$ lbs and $Q = 2212$ lbs. are hung by means of a perfectly flexible inextensible rope over a smooth pulley. Disregarding all friction and the mass of pulley and rope, how long will it take for each mass to move through $s = 5$ ft. from rest?* $(g = 32.)$

Ans. The student should carefully compare this with the two preceding examples.



The tension on the descending side is $P(g-f)$, on the ascending side $Q(g+f)$, where f is the acceleration. Hence

$$P(g-f) = Q(g+f), \text{ or } f = \frac{(P-Q)g}{P+Q} = \frac{224}{1113} \text{ ft.-per-sec. [per sec.}$$

Or we may obtain the same result as follows: The weight of P is Pg poundals. The weight of Q is Qg poundals. The *moving force* is $Pg - Qg$ or $(P-Q)g$ poundals, or the weight of 28 lbs. as in Ex. 11 and 12. The mass moved is $P+Q$. Hence

$$(P+Q)f = (P-Q)g, \quad \text{or} \quad f = \frac{(P-Q)g}{P+Q}.$$

Since $s = \frac{1}{2}ft^2$, we have $5 = \frac{1}{2} \times \frac{224}{1113}t^2$, or $t = 7.04$ sec.

The tension on the rope is $Q(g+f)$ or $P(g-f)$ or $\frac{2QPg}{P+Q}$ poundals, or the weight of $\frac{2QP}{P+Q} = 2225.9$ lbs.

NOTE.—The moving force in Ex. 11, 12, 13 is the weight of 28 lbs. In Ex. 11 the mass moved is $M = 2240$ lbs., hence $28g = Mf$. In Ex. 12 the mass moved is $P + M = 2268$ lbs., hence $28g = (P+M)f$. In Ex. 13 the mass moved is $(P+Q) = 4452$ lbs., hence $28g = (P+Q)f$. In all cases, *moving force = mass moved × acceleration.*

The pressure on the axle is the sum of the two tensions, or $(P+Q)g - (P-Q)f$. If the pulley is not allowed to rotate, the pressure upon the axle would be the weight of P and Q, or $(P+Q)g$. The pressure on the axle during motion is therefore less than when at rest.

(14) *Two equal masses A and B each of M lbs. are suspended by a perfectly flexible inextensible string over a smooth pulley. A small mass C of m lbs. is placed on the mass A. Find the resulting acceleration and the tension of the string, disregarding friction and the mass of the pulley and string.*

Ans. The weight of A and C is $(M+m)g$. The weight of B is Mg. The moving force is then the weight of m or mg. The total mass moved is $2M+m$. We have then by the equation of force

$$mg = (2M+m)f, \quad \text{or} \quad f = \frac{mg}{2M+m}.$$

The mass B rises with this acceleration and the mass A sinks with this acceleration.

The tension on the string, if we consider the mass B, is $M(g+f)$; if we consider the mass A, it is $(M+m)(g-f)$. Substituting the value of f, we have in both cases $T = \frac{2Mg(M+m)}{2M+m}$, or in gravitation units $T = \frac{2M(M+m)}{2M+m}$.

Again, before the mass m is placed on A, the two masses A and B are at rest. The tension added by m is then the force which gives acceleration to the masses A and B, or $2Mf = m(g-f)$, or $f = \frac{mg}{2M+m}$.

(15) *Suppose that in the preceding example the mass m is removed at the end of the time t. Find the resulting motion.*

Ans. The velocity at the end of the time t is $v = ft = \frac{mgt}{2M+m}$, and the space through which A has fallen or B risen in that time is

$$s = \frac{1}{2}ft^2 = \frac{mgt^2}{2(2M+m)}.$$

When m is removed the moving force is zero, and hence the acceleration is zero and the masses A and B continue to move with the uniform velocity $v = \frac{mgt}{2M+m}$.

(16) *A man in an elevator at rest holds in his hand a spring-balance with a mass of* 1 *lb. suspended from it, the balance therefore reading* 1 *lb. Upon starting he observes the balance to register* 15 *oz. for a certain time. Then it changes to* 17 *oz. for a certain time. Then it again registers* 1 *lb., and he finds the elevator at rest and that it has descended* 128 *ft. Find the time of descent.* ($g = 32$.)

Ans. When the balance registers 15 oz. we have $g - f_1 = \frac{15}{16}g$, or the constant acceleration $f_1 = +2$ ft.-per-sec. per sec. In the time t_1, then, during which the balance registers 15 oz. the velocity increases from 0 to $v_1 = f_1t_1$ ft. per sec., and the space passed over is $s_1 = \frac{v_1}{2}t_1 = \frac{1}{2}f_1t_1^2$. When the balance registers 17 oz. the elevator is *retarded*, and we have $g + f_2 = \frac{17}{16}g$ or $f_2 = 2$ ft.-per-sec. per sec. Since the retardation is equal then to the acceleration, the elevator comes to rest in the same time, t_1, which it required to attain its velocity v_1, and passes over the same distance while coming to rest. We have then the space passed over $128 = 2s_1 = f_1t_1^2$, or $t_1 = 8$ sec. The total time is then $2t_1 = 16$ sec.

(17) *An elevator whose mass is* 2240 *lbs. is descending a shaft which is* 100 *feet deep. The chain by which it is suspended has a uniform tension of* 1680 *lbs. If the elevator starts from rest at the top of the shaft, with what velocity would it reach the bottom? Disregard the mass of the chain.*

Ans. We have $2240(g - f) = 1680g$. If $g = 32$ ft.-per sec. per sec., the acceleration $f = 8$ ft.-per-sec. per sec. Since $v = \sqrt{2fs}$ (page 51, Vol. I, *Kinematics*), we have $v = \sqrt{1600} = 40$ ft. per sec.

(18) *Suppose in the preceding example the elevator is counterbalanced by a mass at the other end of the chain, the chain passing over a pulley. Neglecting friction and mass of the chain, what must this mass be in order that the tension may be* 1680 *lbs.? The chain is supposed perfectly flexible and inextensible.*

Ans. $m(g + 8) = 1680g$, or taking $g = 32$ ft.-per-sec. per sec., $m = 1344$ lbs.

(19) *A man can lift* 168 *lbs. on the ground. If in an elevator descending with an acceleration of* 8 *ft.-per-sec. per sec., how much can he lift? If ascending with the same acceleration, how much can he lift?*

Ans. $m(g - f) = 168g$, or taking $g = 32$ ft.-per-sec. per sec., $m = 224$ lbs. If ascending, $m = 134.4$ lbs.

(20) *A rope passes over a pulley and has a mass P lbs. attached at one end and on the other side a mass Q lbs. which slides along the rope. If P remains at rest, find the friction which must act between the rope and Q; also the acceleration of Q.*

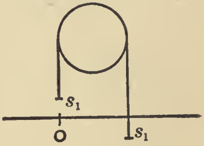
Ans. The friction must equal Pg poundals. The force acting on Q to move it is $(Qg - Pg)$ poundals. If f is the acceleration of Q, then

$$Qg - Pg = Qf, \quad \text{or} \quad f = \frac{(Q-P)g}{Q}.$$

Hence the velocity at the end of the time t, starting from rest of Q, is $v = ft = \dfrac{(Q-P)gt}{Q}$, and the distance described is $s = \dfrac{1}{2}ft^2 = \dfrac{(Q-P)gt^2}{2Q}$.

(21) [*If a perfectly flexible and perfectly smooth rope is hung over a perfectly smooth pin, find the time it will run itself off.*

Let $2l =$ the length of the rope, $m =$ the mass of a unit of length of the rope. Then the mass of the rope is $2ml$.



Let motion commence when one end is at a distance of $2s_1$ above the other. Take the origin of co-ordinates half way between these two ends at O.

At any other instant after motion has commenced let the distance of the short end above O be s. Then the unbalanced mass at that instant is $2ms$, and the moving force at that instant is $2mgs$. Since the mass moved by this force is $2ml$ and the acceleration is $\dfrac{d^2s}{dt^2}$, we have

$$2mgs = 2ml\frac{d^2s}{dt^2}, \quad \text{or} \quad \frac{d^2s}{dt^2} = \frac{gs}{l}.$$

Multiply by ds and integrate and we have

$$\frac{ds^2}{dt^2} = v^2 = \frac{gs^2}{l} + C.$$

When $s = s_1$ let $v = 0$. Then $C = -\dfrac{gs_1{}^2}{l}$ and

$$\sqrt{\frac{g}{l}}\,.\,dt = \frac{ds}{\sqrt{s^2 - s_1{}^2}}.$$

Integrating again, and we have

$$\sqrt{\frac{g}{l}}\,.\,t = \text{logn}\,(s + \sqrt{s^2 - s_1{}^2}) + C.$$

When $s = s_1$ let $t = 0$. Then $C = -\log s_1$ and

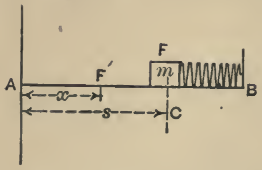
$$t = \sqrt{\frac{l}{g}}\,\text{logn}\,\frac{s + \sqrt{s^2 - s_1{}^2}}{s_1}.$$

When $s = l$, the rope will run off and the time required is

$$t = \sqrt{\frac{l}{g}}\,\text{logn}\,\frac{l + \sqrt{l^2 - s_1{}^2}}{s_1}.$$

(22) [*Suppose a spring whose natural length is AB to be fixed at the end B and compressed from A to C, where it presses against a body of mass m which is perfectly free to move horizontally. Disregarding the mass of the spring, discuss the motion.*

Ans. Let the force at any distance x from A be F', and at the distance $AC = s$ be F.



Then we have $F' : x :: F : s$, or $F' = F\frac{x}{s}$.

The acceleration at any point is $\frac{d^2x}{dt^2}$, and therefore we have

$$m\frac{d^2x}{dt^2} = -F\frac{x}{s},$$

where we take the minus sign because the force and therefore the acceleration is towards the origin (page 50, Vol. I, *Kinematics*).

Since the acceleration is proportional to the distance, the motion is *harmonic* and the integration will give the same results as page 106, Vol. I, *Kinematics*. We have simply to substitute $\frac{F}{ms}$ for $\frac{a'}{r'}$ and s for r and x for s in the equations of page 106, Vol. I, *Kinematics*.

We thus have for the velocity at any distance x from A

$$v^2 = \frac{F}{ms}(s^2 - x^2).$$

For the time of reaching any point distant x from A

$$t = \sqrt{\frac{ms}{F}} \cos^{-1}\frac{x}{s}.$$

For the time of reaching A

$$t = \frac{\pi}{2}\sqrt{\frac{ms}{F}}.$$

For the velocity at A

$$v = s\sqrt{\frac{F}{ms}}.$$

If F is given in pounds, or gravitation measure, it must be multiplied by g whenever it occurs.

We have then, taking F in gravitation units, for the time of reaching A,

$$t = \frac{\pi}{2}\sqrt{\frac{ms}{Fg}},$$

or half the time of vibration of a pendulum whose length is $\frac{m}{F}s$ (page 275, Vol. I).

If the spring is attached to the mass m, the mass will vibrate to a distance s on the other side of A, the entire time of vibration being that of a pendulum of length $\frac{m}{F}s$, and the motion will then be periodic.

If the spring is not attached to the mass m, the mass will leave the spring at A, and move with uniform velocity

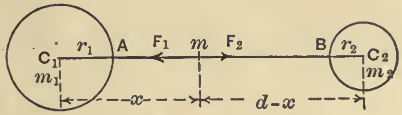
$$v = s\sqrt{\frac{Fg}{ms}},$$

where F is in gravitation units.

If the spring were destitute of mass it would stop at A. But as it has mass, it will pass A and vibrate back and forth also.

(23) [*Discuss the motion of a particle under the mutual attraction of two spherical bodies.*

Ans. Let the mass of the two spherical bodies be m_1 and m_2, and their radii r_1 and r_2 respectively.



Let the particle of mass m be at the distance x from the centre of mass C_1 of m_1, and let the distance C_1C_2 between the centres of mass of m_1 and m_2 be d, so that the distance of the particle from the centre of mass of m_2 is $d - x$.

From page 46, Vol. II, *Statics*, the spheres attract any particle m between them as if the entire mass of each sphere were concentrated at its centre of mass.

Let f_1 and f_2 be the accelerations due to attraction at the surfaces A and B of each sphere. Then the weight of the particle m at A would be mf_1, and at B, mf_2.

When the particle is at the distance x from C_1, let the force of attraction upon it due to m_1 be F_1, and that due to m_2 be F_2. Then, since according to Newton's law (page 44, Vol. II, *Statics*) the force of attraction is inversely as the square of the distance, we have

$$F_1 : mf_1 :: r_1^2 : x^2, \qquad \text{or} \quad F_1 = \frac{mf_1r_1^2}{x^2};$$

$$F_2 : mf_2 :: r_2^2 : (d-x)^2, \quad \text{or} \quad F_2 = \frac{mf_2r_2^2}{(d-x)^2}.$$

Let F_2 be greater than F_1. Then the resultant force of attraction towards C_2 upon the particle is

$$F_2 - F_1 = \frac{mf_2r_2^2}{(d-x)^2} - \frac{mf_1r_1^2}{x^2}.$$

If we divide this force by m, we have the acceleration of the particle m

$$\frac{dv}{dt} = \frac{f_2r_2^2}{(d-x)^2} - \frac{f_1r_1^2}{x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiply both sides of (1) by $2dx$. Then, since $\frac{dx}{dt} = v$, we have

$$2vdv = \frac{2f_2r_2^2dx}{(d-x)^2} - \frac{2f_1r_1^2dx}{x^2}.$$

Integrating, we have for the velocity v at the distance x from C_1

$$v^2 = \frac{2f_2r_2^2}{d-x} + \frac{2f_1r_1^2}{x} + \text{Const.}$$

Let $v = v_1$ when $x = r_1$. Then

$$\text{Const.} = v_1^2 - \frac{2f_2r_2^2}{d-r_1} - \frac{2f_1r_1^2}{r_1}.$$

We have then, substituting this value for the constant of integration, for the velocity v at any point distant x from C_1, if v_1 is the initial velocity at A,

$$v^2 = v_1^2 + 2f_2r_2^2\left[\frac{1}{d-x} - \frac{1}{d-r_1}\right] + 2f_1r_1^2\left[\frac{1}{x} - \frac{1}{r_1}\right]. \quad . \quad . \quad . \quad (2)$$

Let v be zero when $x = d_0$. Then we have

$$0 = v_1^2 + 2f_2r_2^2\left[\frac{1}{d-d_0} - \frac{1}{d-r_1}\right] + 2f_1r_1^2\left[\frac{1}{d_0} - \frac{1}{r_1}\right]. \quad . \quad . \quad (3)$$

From (3) we can find the distance d_0 from C_1 for which the velocity v is zero. We see that this value of d_0 depends upon the initial velocity v_1 at A. Suppose this initial velocity to be such that the particle comes to rest at the point for which F_1 and F_2 are equal. Then we have

$$\frac{mf_1r_1^2}{d_0^2} = \frac{mf_2r_2^2}{(d-d_0)^2}, \quad \text{or} \quad f_1r_1^2 = \frac{f_2r_2^2d_0^2}{(d-d_0)^2}.$$

Substituting this value of $f_1r_1^2$ in (3), we have for the corresponding value of v_1

$$v_1^2 = 2f_2r_2^2\left[\frac{1}{d-r_1} - \frac{1}{d-d_0} + \frac{d_0^2}{(d-d_0)^2}\left(\frac{1}{r_1} - \frac{1}{d_0}\right)\right]. \quad . \quad . \quad (4)$$

If v_1 is less than the value given by (4), the particle will come to rest before it reaches d_0 and will then return to A. If v_1 is greater than the value given by (4), the particle will pass the point of equilibrium at d_0 and then fall towards B.

We can simplify (4) as follows: When $F_1 = F_2$, we have

$$\frac{d_0^2}{(d-d_0)^2} = \frac{f_1r_1^2}{f_2r_2^2}.$$

The point at which the attractions F_1 and F_2 are equal divides the distance d then into two portions d_0 and $d - d_0$, whose ratio n is given by

$$n = \frac{d_0}{d-d_0} = \frac{r_1}{r_2}\sqrt{\frac{f_1}{f_2}}.$$

Hence

$$d_0 = \frac{nd}{1+n} \quad \text{and} \quad d - d_0 = \frac{d}{1+n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We have also by Newton's law (page 44, Vol. II, *Statics*)

$$mf_1 = \kappa\frac{mm_1}{r_1^2}, \quad mf_2 = \kappa\frac{mm_2}{r_2^2}.$$

Hence

$$\frac{f_1}{f_2} = \frac{m_1r_2^2}{m_2r_1^2}, \quad \text{or} \quad \frac{m_1}{m_2} = \frac{f_1r_1^2}{f_2r_2^2}, \quad \text{or} \quad n = \sqrt{\frac{m_1}{m_2}}. \quad . \quad . \quad . \quad . \quad (6)$$

If we insert in (4) the values of d_0 and $d - d_0$ given by (5), we obtain for v_1

$$v_1 = \sqrt{2f_2r_2}\sqrt{\frac{r_2}{d-r_1} - \frac{r_2}{d}(1+n)^2 + n^2\frac{r_2}{r_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where n is given by (6).

If v_1 is less than this, the particle will come to rest before it reaches d_0 and will then return to A. If v_1 is greater than this, the particle will pass the point of equilibrium at d_0 and then fall towards B.

From (5) and (6) we have for the distance d_0 to the point of equilibrium

$$d_0 = \frac{d}{1 + \sqrt{\frac{m_2}{m_1}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

CHAPTER II.

DEFLECTING FORCE.

DEFLECTING FORCE. SIMPLE CONICAL PENDULUM. DEFLECTING FORCE ON THE EARTH. DEFLECTING FORCE—PARTICLE MOVING ON EARTH'S SURFACE. DEVIATION OF A FALLING BODY DUE TO EARTH'S ROTATION.

Deflecting Force.—We have seen (page 83, Vol. II, *Statics*) that whatever the motion of a body, its centre of mass moves as if the entire mass of the body were concentrated at the centre of mass and all the forces were transferred to this point.

We have also the equation of force

$$F = mf.$$

Suppose then the centre of mass of a body to move in a curve whose radius of curvature at any point is ρ. Then (page 76, Vol. I, *Kinematics*), the tangential acceleration is $f_t = \rho\alpha$ and the normal acceleration is $f_n = \frac{v^2}{\rho} = \rho\omega^2$, where α is the rate of change of angular speed, v the linear speed and ω the angular speed at the point.

If m is the mass of the body, we have then acting at the centre of mass the *tangential force*

$$f_t = mf_t = m\rho\alpha,$$

and the *normal force*

$$F_n = mf_n = \frac{mv^2}{\rho} = m\rho\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This latter force is the **deflecting force**, because it is at right angles to the direction of motion and *can therefore cause no change of speed, but only change of direction of motion.*

If the path is a circle, ρ is constant and equal to the radius r, and we have

$$F_n = \frac{mv^2}{r} = mr\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If there is no tangential force, there will be no change of speed. In such case, if T is the *periodic time*, or the time of a complete revolution, we have $\omega = \frac{2\pi}{T}$, or $v = \frac{2\pi r}{T}$, and

$$F_n = \frac{4\pi^2 mr}{T^2} = mr\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the force is required in gravitation units, we must divide by g (page 6, Vol. II, Statics).

The force F_n in the case of circular motion is sometimes called "*centripetal force*," since it constantly draws the particle towards the fixed centre. This term is correct and descriptive.

The more general term is, however, *deflecting force*, since it constantly draws the particle out of the tangent to its path at any instant without affecting the speed, the centre not being fixed in general.

It is also often called "*centrifugal force*," that is, force *away* from the centre. This is neither correct nor descriptive. The same force evidently cannot be both centripetal and centrifugal, and as there is really no force on the body acting away from the centre and there really is a force acting on the body towards the centre, the term centrifugal is incorrect when the body is considered.

A particle moving in a circle was formerly thought to possess an inherent so-called "centrifugal force," by virtue of which it tended to fly away from the centre, and hence the term, which has unfortunately passed into common use.

But a body cannot change its direction of motion. The centre of mass in circular motion is moving at any instant in a straight line tangent to the path, and an external force is necessary to make the direction of the motion change.

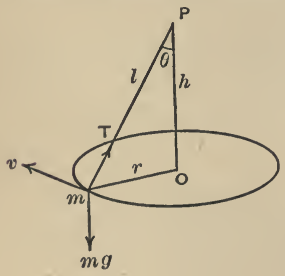
This external force must be in the direction of the acceleration or towards the centre of the circle. It is therefore a deflecting force or "centripetal" force.

If the body is attached by a string to the centre of the circle we have *tensile stress* in the string, and two equal and opposite stresses of action and reaction (page 37) between the body and centre of the circle. If now we consider the body from the standpoint of the centre, the force on it is the centripetal stress towards the centre. This is the deflecting force.

If we *consider the centre* from the standpoint of the body, the force on it is the centrifugal stress away from the centre. This is the so-called "centrifugal force."

The term "centrifugal force" always signifies then the opposite aspect of the stress between the centre and body. In this sense it may properly be used, but as it is apt to be misleading it is just as well to discard it altogether.*

Simple Conical Pendulum.—The simple conical pendulum consists of a particle of mass m attached to a fixed point P by a massless inextensible string of length l, and moving with uniform speed v in a circular path about a vertical axis through the fixed point.



In this case the particle is acted upon by two forces, its weight mg vertically downwards and the tension T of the string directed towards the fixed point P. If the particle moves with uniform speed v in the circle whose radius is $r = l \sin \theta$, where θ is the inclination of the string to the vertical, the vertical component $T \cos \theta$ of the tension T must balance the weight mg, and the horizontal component $T \sin \theta$ of the tension T must be equal to the deflecting force $\frac{mv^2}{r}$ necessary to make the particle move in a circle with uniform speed. We have then

---

* "When I was about nine years old, I was taken to hear a course of lectures by an itinerant lecturer in a country town, to get as much as I could of the second half of a good, sound philosophical omniscience. . . . .

"'You have heard what I have said of the wonderful centripētal force, by which Divine Wisdom has retained the planets in their orbits round the sun. But, ladies and gentlemen, it must also be clear to you that if there were no other force in action, this centripētal force would draw our earth and the other planets into the sun, and universal ruin would ensue. To prevent such a catastrophe, the same Wisdom has implanted a centrifūgal force of the same amount, and directly opposite.' . . . .

"I had never heard of Alfonso X. of Castile, but I ventured to think that if Divine Wisdom had just let the planets alone it would come to the same thing with equal and opposite troubles saved."—De Morgan, *Budget of Paradoxes.*

$$mg - T \cos \theta = 0, \quad \text{or} \quad T \cos \theta = mg, \quad . \quad . \quad . \quad . \quad (1)$$

and from equation (2), page 15,

$$T \sin \theta = \frac{mv^2}{r} = \frac{mv^2}{l \sin \theta} = mr\omega^2 = ml \sin \theta . \omega^2, \quad . \quad . \quad . \quad (2)$$

where ω is the angular speed.

We can find T from either (1) or (2). Squaring (1) and (2) and adding, we have also

$$T = m\sqrt{g^2 + \frac{v^4}{r^2}} = m\sqrt{g^2 + r^2\omega^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also dividing (2) by (1), we have

$$\tan \theta = \frac{v^2}{gr} = \frac{v^2}{gl \sin \theta} = \frac{r\omega^2}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For T in gravitation units we must divide (3) by g as usual.

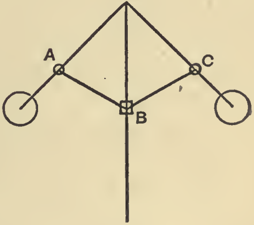
Let h be the distance of the fixed point P above the plane of motion, or the height of the pendulum. Then since $h \tan \theta = r$, we have from (4)

$$v^2 = \frac{r^2 g}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If then ω is the angular speed of the particle about the centre O, $r\omega = v$, and from (5) $\omega = \sqrt{\frac{g}{h}}$. If t is the time of a revolution,

$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This is the same as the time of oscillation of a simple pendulum of length h (page 154, Vol. I, *Kinematics*).



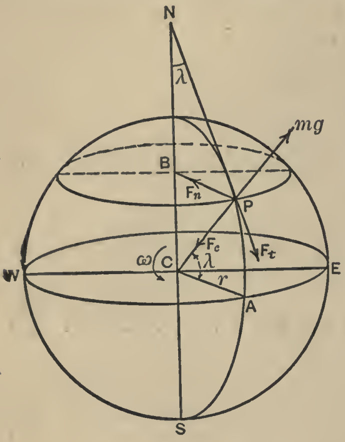
COR. 1. If θ is indefinitely small, h and l are equal and

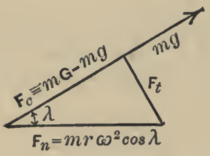
$$t = 2\pi\sqrt{\frac{l}{g}},$$

and we have the case of the simple pendulum.

COR. 2. Since $\omega = \sqrt{\frac{g}{h}}$, we see that the greater the angular velocity the less h, and, as l is constant, the greater r. This fact is taken advantage of in the *steam-engine governor*. As the piston speed increases, the spindle PO revolves more rapidly, the balls separate and the slide at B rises and by means of levers acts upon the valves of the engine to diminish the supply of steam.

Deflecting Force on the Earth.—Suppose the earth to be a homogeneous sphere of radius r and centre C. Let WAE represent the equator, NS the axis, and let P be any point of the surface on the meridian PAS, so that the latitude of P is $\lambda = PCA$.





Let a particle of mass m rest on the earth's surface at P. Let G be the acceleration of gravity acting towards the centre of mass C along the radius PC. Then mG is the force of attraction. Let g be the *observed* acceleration of gravity at the point P. Then the observed weight of the particle at P as given by a spring-balance is mg acting towards the centre C along the radius PC. The pressure of the earth upon the particle is then mg acting in the opposite direction.

Then the difference between mG and mg, or $mG - mg$, is that portion of the earth's attraction which is required to keep the particle in contact with the earth. This force has no effect upon a spring-balance. We call it the *central deflecting force* and denote it by F_c. We have then

$$F_c = mG - mg. \quad . \quad . \quad (1)$$

We can resolve this central deflecting force F_c into two components, one F_t tangent to the meridian at P and one F_n in the direction PB. This latter component is the deflecting force on the particle necessary to keep it on the latitude circle whose radius is $BP = r \cos \lambda$. We have then (page 15), since the angular velocity $\omega = \frac{2\pi}{T}$, where T is the time of rotation,

$$F_n = mr\omega^2 \cos \lambda = \frac{4\pi^2 mr}{T^2} \cos \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have therefore, since $\cos \lambda \sin \lambda = \frac{1}{2} \sin 2\lambda$,

$$F_c = F_n \cos \lambda = mr\omega^2 \cos^2 \lambda = \frac{4\pi^2 mr}{T^2} \cos^2 \lambda ; \quad . \quad . \quad . \quad (3)$$

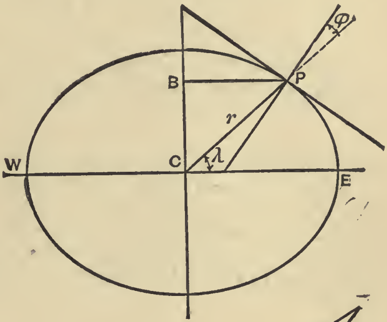
$$F_t = F_n \sin \lambda = \frac{1}{2} mr\omega^2 \sin 2\lambda = \frac{2\pi^2 mr}{T^2} \sin 2\lambda. \quad . \quad . \quad . \quad (4)$$

From (1) and (3) we have

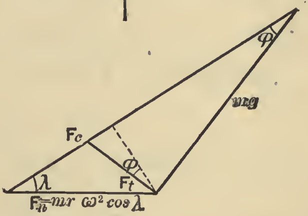
$$mg = mG - mr\omega^2 \cos^2 \lambda = mG - \frac{4\pi^2 mr}{T^2} \cos^2 \lambda. \quad . \quad . \quad . \quad (5)$$

Hence, *the central deflecting force F_c is proportional to the square of the cosine of the latitude and diminishes the observed force of gravity at the surface.*

If the earth were a homogeneous sphere the effect of the tangential component F_t upon liquid particles on the surface would be to force them towards the equator and thus increase the equatorial and diminish the polar diameter. The fact that the earth is not a sphere thus indicates that the now solid portions may once have existed in a plastic condition. The equatorial diameter is found to exceed the polar by about 26 miles. The ratio of this difference to the equatorial diameter, called the **ellipticity** of the earth, is about $\frac{1}{300}$. The earth is considered then as an ellipsoid of revolution with this ellipticity, so that the direction of the observed force of gravity or of the plumb-line AP (see following figure) is *always normal to the surface of the earth or to a water-level* and hence does not pass through the centre of figure C except at the equator and the poles.



The force of attraction mG is then resolved into two components, one mg normal to the surface and one F_n along BP. This latter is the deflecting force necessary to keep the particle on the latitude circle whose radius is $BP = r \cos \lambda$. The former is balanced by the pressure of the earth upon the particle. There is then no resultant tangential force F_t and no tendency of the particle at P to move towards the equator.



Let r be the radius vector for any point P whose latitude is $PCE = \lambda$, and let ϕ be the angle of the normal AP to the surface with r. Then we have for the earth regarded as a spheroid

$$F_c = mG - \frac{mg}{\cos \phi}, \quad \quad (1)$$

and as before

$$F_n = mr\omega^2 \cos \lambda. \quad \quad (2)$$

We have then from the figure

$$F_c = F_n \cos \lambda - F_n \sin \lambda \tan \phi = mr\omega^2 \cos \lambda\,(\cos \lambda - \sin \lambda \tan \phi); \quad (3)$$

$$F_t = \frac{F_n \sin \lambda}{\cos \phi} = \frac{1}{2} mr\omega^2 \frac{\sin 2\lambda}{\cos \phi}; \quad \quad (4)$$

and from (1) and (3)

$$mg = mG \cos \phi - mr\omega^2 \cos \lambda \cos \phi\,(\cos \lambda - \sin \lambda \tan \phi). \quad (5)$$

We have also the tangential component of the pressure mg equal and opposite to F_t, so that there is no resultant tangential force. Since in the case of the earth the deviation from a sphere is small, the angle ϕ is very small, and equations (1) to (5) reduce to equations (1) to (5), page 18. We can then practically *treat the earth as a sphere of mean radius r and neglect the tangential component F_t.*

COR. 1. If we take the mean radius $r = 3960$ miles and $T = 86164$ seconds, we have

$$\frac{4\pi^2 r}{T^2} = r\omega^2 = 0.111255 \text{ ft.-per-sec. per sec.} \quad . \quad . \quad . \quad . \quad (6)$$

We have then from (5), page 18, for the total acceleration of gravity G at any point P in latitude λ

$$G = g + 0.111255 \cos^2 \lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where g is the observed acceleration of gravity at P.

At the poles $\lambda = 90°$ and $G = g$, or the observed acceleration of gravity at the poles is equal to the total central acceleration G of the earth considered as a homogeneous sphere.

At the equator $\lambda = 0$, and the observed value of g at the equator at sea-level is about 32.09022 ft.-per-sec. per sec. Hence from (7) we obtain

$$G = 32.20148 \text{ ft.-per-sec. per. sec.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The central deflecting force at any point is, from (3), page 18, and (6),

$$F_c = 0.111255 m \cos^2 \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

At the equator we have the central deflecting force, from (9) and (8),

$$F_{cE} = 0.111255 m = \frac{1}{289} mG. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

That is, *the central deflecting force at the equator is about* 1/289 *of the total force of gravity.*

COR. 2. To find the time of rotation T_0 of the earth in order that a particle at any point P may have no weight, i.e. exert no pressure on the surface, we have from (3), page 18, by putting $F_c = mG$ and $T = T_0$,

$$G = \frac{4\pi^2 r}{T_0^2} \cos^2 \lambda, \quad \text{or} \quad T_0 = \sqrt{\frac{4\pi^2 r}{G} \cos^2 \lambda}.$$

But from (10),

$$\frac{4\pi^2 r}{T^2} = \frac{1}{289} G, \quad \text{or} \quad G = \frac{289 \times 4\pi^2 r}{T^2}.$$

Inserting this value of G, we obtain

$$T_0 = \frac{1}{17} T \cos \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

At the equator we have $T_0 = \frac{1}{17} T$. In order, then, that a particle at the equator may have no weight the earth must rotate in one seventeenth of a day.

EXAMPLES.

(1) *A string* 5 *feet long just breaks with a weight of* 20 *lbs. It is fastened to a fixed point at one end, and at the other to a mass of* 5 *lbs. which revolves round the point in a horizontal plane. Find the greatest number of complete revolutions that can be made in a minute without breaking the string.* $(g = 32.)$

Ans. To break the string requires a force of $20g$ poundals. Let n be the number of revolutions per minute. Then $v = \frac{2\pi rn}{60} = \frac{\pi n}{6}$. The stress in the string is $\frac{mv^2}{r} = 20g$, or $v = \sqrt{20g} = \frac{\pi n}{6}$. Hence $n = \frac{6\sqrt{20g}}{\pi} = 48$ complete revolutions.

(2) *Suppose the mass revolves in a vertical plane.*

Ans. Then at the lowest point the stress is $\frac{mv^2}{r} + mg$ poundals, and $\frac{mv^2}{r} + mg = 20g$, or $v = \sqrt{15g} = \frac{\pi n}{6}$. Hence $n = \frac{6\sqrt{15g}}{\pi} = 41$ complete revolutions.

(3) *What portion of their weight do bodies lose at the equator, assuming the radius of the earth* 4000 *miles and* $g = 32$ *ft.-per-sec. per sec.?*

Ans. About $\frac{1}{286}$.

(4) *Find the length of day in order that a body may possess no weight at the equator.*

Ans. About one seventeenth of 24 hours.

(5) *A skater describes a circle of* 100 *feet radius, with a speed of* 18 *feet per second. Find his inclination to the ice.*

Ans. About $84° 13'$.

(6) *An engine of* 20 *tons runs with a speed of* 30 *miles an hour, at a part of the track where the radius of curvature is* 10 *miles. Find the pressure against the rails.*

Ans. $\frac{11}{15g}$ tons.

(7) *What should be the difference of level between the rails when the radius of curvature of a railway curve is* 300 *yards, the breadth of gauge* 4 *ft.* 8½ *in., the greatest speed of a train* 45 *miles per hour?*

Ans. 8.6 inches.

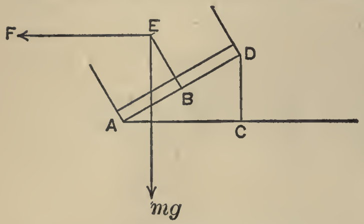
(8) *A mass of* 32 *lbs. moves in a circle of radius* 4 *ft. with a uniform speed of* 20 *ft. per sec. Find the force directed towards the centre.* $(g = 32.)$

Ans. $\frac{3200}{g}$ poundals or 100 lbs.

(9) *A mass of* 32 *lbs. revolves uniformly in a circle of radius* 4 *feet, making* 48 *revolutions per minute. Find the force directed towards the centre.*

Ans. The angular velocity is $\omega = 1.6\pi$ radius per sec. The acceleration is $f = r\omega^2 =$ about 100 ft.-per-sec. per sec. The force is 3200 poundals or about 100 pounds, taking $g = 32$ ft.-per-sec. per sec.

(10) *Find the necessary elevation of the outer rail on a railroad-track on a curve of radius r, so that an engine weighing m lbs. moving with a speed v can pass without lateral pressure on the rails by the wheel-flanges.*



Ans. Let E be the centre of mass of the engine and $h = EB$ the distance of the centre of mass above the rails.

We have acting at E the weight mg vertical. This can be resolved itno a component along EB and a horizontal component EF. If there is no flange pressure the component EF must be equal to the deflecting force $\frac{mv^2}{r}$. Let α be the angle of elevation DAC. Then $EF = \frac{mv^2}{r} = mg \tan\alpha$ or $\tan\alpha = \frac{v^2}{gr} = \frac{DC}{AC}$. If α is small, AC is practically equal to AD = the gauge of track. Hence approximately,

$$DC = \frac{ADv^2}{gr}.$$

For the standard gauge of $AD = 4$ ft. $8\frac{1}{2}$ in. $= 56.5$ inches, taking $g = 32$ ft.-per-sec. per sec. and v in ft. per sec. and r in feet, we have

$$DC = \frac{7v^2}{4r} \text{ inches, nearly.}$$

If v is taken in miles per hour and r in feet, we have

$$DC = \frac{15v^2}{4r} \text{ inches, nearly.}$$

(11) *Find the speed v of an engine on a curved level track of radius r and gauge w when it is just on the point of overturning, the centre of mass being h above the rails.*

Ans. $v = \sqrt{\frac{gwr}{2h}}$. If h is 6 ft. and w is 4 ft. $8\frac{1}{2}$ in., we have, taking $g = 32$ ft.-per-sec. per sec., the speed $v = 3.55\sqrt{r}$ ft. per sec., nearly, where r must be taken in feet.

(12) *Masses of 3 and 4 lbs. are fixed at the ends of a horizontal rod. If the rod is without mass, find the position of the vertical axis round which it must revolve in order that the deflecting forces may be equal.*

Ans. At $\frac{3}{7}l$ from the larger mass.

(13) *A particle of m lbs. is fastened to a fixed point by a string l ft. long, and describes a horizontal circle so that the string generates a right circular cone of vertical angle 2θ. Show that the speed of the particle is $v = \sqrt{lg \sin^2\theta \sec\theta}$.*

(14) *Find the height h of a governor to run at 60 revolutions per minute.* ($g = 32$.)

Ans. $h = 9.72$ inches.

(15) *One end of a string $2l$ feet long is fastened to a point A on a smooth vertical rod, the other to a small ring P of mass m lbs. which slides on the rod. Another mass Q of m' lbs. is fastened to the middle point of the string and revolves with a velocity v ft. per sec. in a horizontal circle, so that the angle AQP is a right angle. Show that* $v^2 = \dfrac{lg(m' + 2m)}{\sqrt{2}\, m'}$.

(16) *What ought to be the difference of level between the rails, when the radius of curvature of a railway curve is 300 yards, the breadth of gauge 4 ft. 8½ in., and the highest speed of the train 45 miles an hour?*

Ans. 8.4 inches.

(17) *A railway car is going round a curve of 500 ft. radius at 30 miles an hour. Find how much a plummet hung from the roof by a thread 6 ft. long would be deflected from the vertical if there were no pressure on the rails by the wheel-flanges.*

Ans. About 0.72 ft.

(18) *A particle is whirled round horizontally by a string 2 yards long. What is the time of one revolution when the tension of the string is 4 times the weight of the particle?*

Ans. $T = \pi\sqrt{\dfrac{r}{g}} = 1.35$ sec.

(19) *A man standing at one of the poles of a rotating planet whirls a body of 20 lbs. on a smooth horizontal plane by a string 1 yard long at the rate of 100 turns per minute. He finds that the difference of the forces which he has to exert according as he whirls the body one way or the other is 0.01 lb. Find the period of rotation of the planet.*

Ans. 13 hr. 37 min. 21.6 sec.

(20) *A railroad car is going round a curve of 500 ft. radius at the rate of 30 miles an hour. Find how much a plummet hung from the roof by a thread will be deflected from the vertical.*

Ans. 6° 51′.4.

(21) *A particle of mass m attached by an inextensible string of length l to a fixed point moves in a vertical plane in a circle of radius r. Find the tension T in any position, the least and greatest values of T. Show also that the least value of the speed with which a circle will be described is $\sqrt{lg}$, and that when the speed has this value the greatest value of T is equal to six times the weight of the particle.*

Ans. Let A be the highest point of the path, and P any point, and θ the angle subtended by the arc AP. Let V be the speed at A, and v that at any point P. The normal acceleration is $\dfrac{v^2}{l}$. The vertical acceleration is g. The vertical distance fallen from A is $l(1 - \cos\theta)$. Hence the velocity at any point is given by

$$v^2 = V^2 + 2gl(1 - \cos\theta),$$

and the normal acceleration is

$$\frac{v^2}{l} = \frac{V^2}{l} + 2g(1 - \cos\theta).$$

The normal component of g is $g\cos\theta$. The deflecting force then is

$$T + mg\cos\theta = \frac{mv^2}{l} = \frac{mV^2}{l} + 2gm(1 - \cos\theta).$$

Hence

$$T=\frac{mV^2}{l}+2gm-3gm\cos\theta.$$

The greatest value of T will be when $\theta=180°$, or when the particle is at the lowest point, and equal to

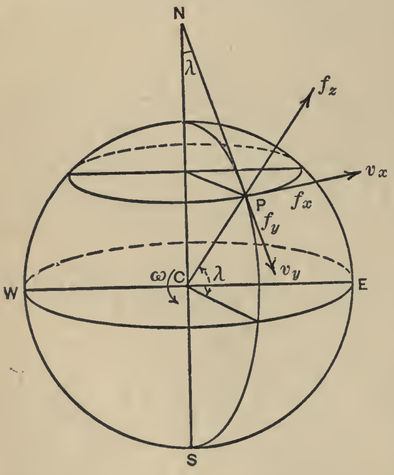
$$T=\frac{mV^2}{l}+5mg.$$

The least value of T will be when $\theta=0$, or when the particle is at the highest point, and equal to

$$T=\frac{mV^2}{l}-mg.$$

Placing this equal to zero, we find for the least value of V with which a circle will be described $V=\sqrt{lg}$. Substituting this in the preceding value of T, we find $T=6mg$.

Particle Moving on the Earth's Surface.—Let the particle P instead of being at rest on the surface of the earth have a velocity v relative to the earth at any instant in any direction tangent to the earth's surface.



Take the point P as origin, the axis of X towards the east, the axis of Y towards the north, the axis of Z along the radius through P.

Let v_x and v_y be the components of v along the axes of X and Y, so that v_x is positive towards the east and negative towards the west, and v_y is positive towards the north and negative towards the south.

Let $P_1P_2=v_yt$ be the distance south along the meridian described by P in north latitude, *in an indefinitely small time t.* If there were no rotation, P_1P_2 would coincide with the meridian through P_1. But owing to the rotation of the earth this meridian moves to $P_1'M$, while P_1 moves to P_2', so that if $P_1'O'$ is parallel to the axis NO, the angle $MO'P_2'=\omega t$, where ω is the angular velocity of rotation. The angle $O'P_1'P_2'=\lambda=$ the latitude of P_1. We have then $O'P_2'=v_yt\sin\lambda$ and

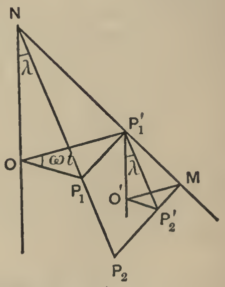
$$MP_2'=v_yt\sin\lambda\,.\,\omega t.$$

But if f_x is the acceleration due to rotation of P with reference to the meridian M, we have

$$MP_2'=\frac{1}{2}f_xt^2=v_y\omega t^2\sin\lambda.$$

Hence we have for the acceleration of P *with reference to the meridian M*, due to rotation and the velocity v_y,

$$f_x=2\omega v_y\sin\lambda.\quad\ldots\ldots\ldots\quad(1)$$



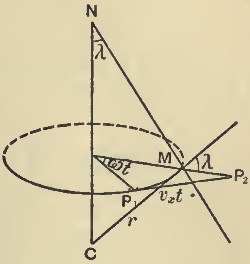
Equation (1) is general if we take v_y positive towards the north and negative towards the south, and λ positive or negative according as the point is north or south of the equator. Thus in the figure v_y is south or negative, and hence for north latitude f_x is negative or towards the west. (Compare page 216, Vol. I, *Kinematics*.)

Again, let $P_1P_2 = v_xt$ be the distance east described by P in north latitude in an indefinitely small time t. The meridian moves to M while P_1 moves to P_2, so that the angle $MP_1P_2 = \omega t$. We have then

$$MP_2 = v_xt \,.\, \omega t,$$

and its projection on the meridian is $MP_2 \sin\lambda = \omega v_xt^2 \sin\lambda$ towards the south. If f_y is the acceleration of P with reference to the meridian, due to rotation and the velocity v_x, we have

$$\frac{1}{2}f_yt^2 = -\,\omega v_xt^2 \sin\lambda, \text{ or } f_y = -\,2\omega v_x \sin\lambda. \quad (2)$$



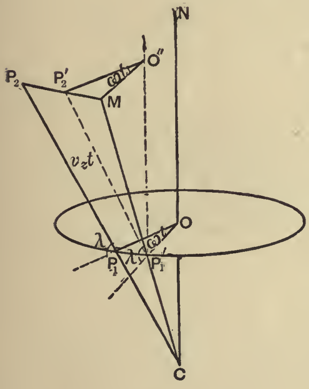
Equation (2) is general if we take v_x positive towards the east, negative towards the west, latitude north positive, south negative, and f_y positive towards the north, negative towards the south.

Again, we have in the preceding figure for the projection of MP_2 along the radius r, $MP_2 \cos\lambda = \omega v_xt^2 \cos\lambda$ upwards. If f_z is the radial acceleration due to rotation and the velocity v_x of P with reference to the meridian, we have

$$\frac{1}{2}f_zt^2 = \omega v_xt^2 \cos\lambda, \quad \text{or} \quad f_z = 2\omega v_x \cos\lambda. \quad . \quad . \quad . \quad . \quad (3)$$

But we also have the acceleration of P with reference to the meridian when there is no rotation, $\frac{v^2}{r} = \frac{v_x^2 + v_y^2}{r}$ due to the velocity v, and also the downward acceleration g due to gravity. Hence the total radial acceleration of P *with reference to the meridian*, due to rotation and the velocities v_x and v_y, is

$$f_z = -\,g + \frac{v_x^2 + v_y^2}{r} + 2\omega v_x \cos\lambda. \quad . \quad . \quad . \quad . \quad . \quad (4)$$



Equation (4) is general if we take v_x positive towards the east, negative towards the west, and v_y positive towards the north, negative towards the south, λ positive for north, negative for south latitude, and f_z positive upwards, negative downwards.

Deviation of a Falling Body by Reason of the Rotation of the Earth.—Let a particle be projected upwards along the radius of the earth with a relative velocity v_z, and let $P_1P_2 = v_zt$ be the distance described in an indefinitely small time t. If there were no rotation, P_1P_2 would coincide with the radius through P_1. But owing to the rotation of the earth the meridian moves to $P_1'M$ while P_1 moves to P_2', so that if P_1O' is parallel to the axis NO, the angle

$MO'P_2' = \omega t$, where ω is the angular velocity of rotation. The angle $O'P_1'P_2' = 90 - \lambda$, where λ is the latitude of P_1. We have then $O'P_2' = v_z t \,.\, \cos\lambda$ and

$$MP_2' = -\, v_z t \cos\lambda \,.\, \omega t.$$

But if f_x is the acceleration due to rotation of P with reference to the meridian, we have

$$MP_2' = \frac{1}{2} f_x t^2 = -\, v_z t^2 \omega \cos\lambda.$$

Hence we have for the acceleration in longitude of P *with reference to the meridian*, due to rotation and the velocity v_z,

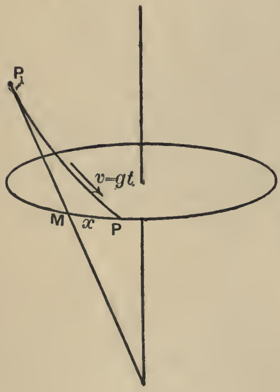
$$f_x = -\, 2\omega v_z \cos\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1) is general if we take v_z positive upwards and negative downwards, north latitude positive, south latitude negative, and f_x positive towards the east, negative towards the west.

For a falling body, then, v_z is negative and we have f_x essentially positive or towards the east. Hence a falling body falls *to the east* of the point vertically beneath it at the start.

Let t be the time of fall. Then if the particle starts from rest, we have for the height of fall

$$h = \frac{1}{2} g t^2, \quad \text{or} \quad t = \sqrt{\frac{2h}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$



The resultant acceleration at any point of the path is then

$$\sqrt{g^2 + f_x^2} = \sqrt{g^2 + 4v^2\omega^2 \cos^2\lambda}.$$

But $\omega = \dfrac{2\pi}{60 \times 60 \times 24}$. For ordinary falls, then, we may neglect $4v^2\omega^2 \cos^2\lambda$, and we have practically the acceleration at any point of the path equal to g. The velocity at any point of the path is then practically $v = gt$, and from (1)

$$f_x = 2\omega g t \cos\lambda. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The mean acceleration is then

$$\frac{1}{2} f_x = \omega g t \cos\lambda,$$

and the final velocity in longitude is then $\frac{1}{2} f_x t = \omega g t^2 \cos\lambda$. The velocity in longitude varies then as the square of the time, or as the ordinate to a parabola. The mean velocity in longitude is then $\frac{1}{3}\omega g t^2 \cos\lambda$, and hence the distance in longitude is

$$x = \frac{1}{3} \omega g t^3 \cos\lambda.$$

Inserting the value of t from (2), we have

$$x = \frac{1}{3} g\omega \cos\lambda \sqrt{\left(\frac{2h}{g}\right)^3} = \frac{2}{3} h\omega \cos\lambda \sqrt{\frac{2h}{g}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) gives the deviation in longitude of a falling body by reason of the earth's rotation. It is always towards the east or positive. For a body projected vertically upwards it is towards the west.

[Equation (4) is deduced by Calculus as follows. From (3)

$$f_x = \frac{d^2x}{dt^2} = 2\omega gt \cos \lambda.$$

Integrating and making $v_x = 0$ when $t = 0$, we have

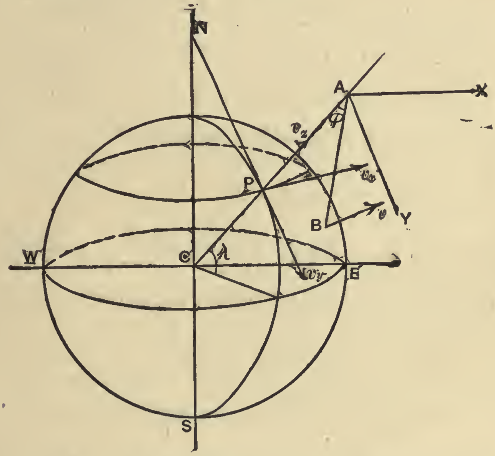
$$v_x = \frac{dx}{dt} = \omega gt^2 \cos \lambda.$$

Integrating again and making $x = 0$ when $t = 0$, we have

$$x = \frac{1}{3}\omega gt^3 \cos \lambda.$$

Substituting the value of t from (2), we obtain (4).

[**Foucault's Pendulum.** — Let a pendulum AB of length l swing in any plane through the vertical APC. Take the point of suspension A as origin, the axis of X towards the east, the axis of Y towards the north, the axis of Z vertical as shown in the figure.



Let the angle of the pendulum at any instant with the vertical be ϕ, and let v_x, v_y, v_z be the components along the axes of its velocity at that instant. Let x, y, z be the co-ordinates of the end B at that instant.

Then the pendulum AB makes angles with the axes of X, Y, Z, given respectively by

$$\cos a = \frac{x}{l}, \quad \cos b = \frac{y}{l}, \quad \cos c = \frac{z}{l}.$$

If we regard the pendulum AB as a simple pendulum, i.e., a massless string with a particle B of mass m at the extremity, the tension of the string, if v is the velocity at any instant,

$$T = mg \cos \phi + \frac{mv^2}{l} = mg \cos \phi + \frac{m(v_x^2 + v_y^2 + v_z^2)}{l}. \quad . \quad (1)$$

The acceleration along the axis of X of the mass B with reference to A due to rotation and the velocities v_y and v_z is, from (1), page 24, and (1), page 26,

$$2\omega v_y \sin\lambda - 2\omega v_z \cos\lambda.$$

We have then for the force on B parallel to the axis of X

$$m\frac{d^2x}{dt^2} = T\frac{x}{l} + 2m\omega(v_y \sin\lambda - v_z \cos\lambda). \quad . \quad . \quad . \quad . \quad (2)$$

The acceleration along the axis of Y with reference to A due to rotation and the velocity v_x is, from (2), page 25,

$$-\,2\omega v_x \sin\lambda.$$

We have then for the force on B parallel to the axis of Y

$$m\frac{d^2y}{dt^2} = T\frac{y}{l} - 2m\omega v_x \sin\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The acceleration along the axis of Z with reference to A due to rotation and the velocity v_x is, from (3), page 25,

$$2\omega v_x \cos\lambda.$$

We have then for the force on B parallel to the axis of Z

$$m\frac{d^2z}{dt^2} = T\frac{z}{l} - mg + 2m\omega v_x \cos\lambda. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

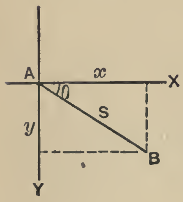
Equations (2), (3) and (4) are the differential equations of the motion of B.

In order to find the motion of the projection of B on a horizontal plane, multiply (2) by y and (3) by x and subtract, and we have, since $v_y = \frac{dy}{dt}$, $v_x = \frac{dx}{dt}$, $v_z = \frac{dz}{dt}$,

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = -\,2\omega \sin\lambda\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) + 2\omega y\frac{dz}{dt}\cos\lambda. \quad . \quad (5)$$

Let s be the distance of the horizontal projection of B from A, so that

$$s = l\cos\phi.$$



Then

$$x^2 + y^2 = s^2,$$

or, differentiating and dividing by dt,

$$x\frac{dx}{dt} + y\frac{dy}{dt} = s\frac{ds}{dt}.$$

Let s make the angle θ with the axis of X. Then

$$x = s\cos\theta, \qquad y = s\sin\theta,$$

and from equation (25), page 84, Vol. I, *Kinematics*, we have, since

$$\frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta = \frac{1}{sdt}d\left(s^2\frac{d\theta}{dt}\right)$$

= the horizontal acceleration perpendicular to s, from (5),

$$\frac{d\left(s^2\frac{d\theta}{dt}\right)}{dt} = -\,2\omega s\frac{ds}{dt}\sin\lambda + 2\omega y\frac{dz}{dt}\cos\lambda. \quad . \quad . \quad . \quad . \quad (6)$$

If we suppose a very long pendulum with small arc of vibration, so that z is very small and can be neglected, equation (6) becomes

$$\frac{d\left(s^2\dfrac{d\theta}{dt}\right)}{dt} = -\,2\omega s\frac{ds}{dt}\sin\lambda.$$

Integrating, we have

$$s^2\frac{d\theta}{dt} = -\,\omega s^2 \sin\lambda + \text{Const.}$$

But $s\dfrac{d\theta}{dt}$ is the horizontal velocity perpendicular to s, and hence $s^2\dfrac{d\theta}{dt}$ is the moment of this velocity with reference to the axis of Z. Let $s = s_1$ at the start, and let this moment be zero when $s = s_1$, that is, let the pendulum swing at the start either in the plane of YZ or XZ. Then we have

$$\text{Const.} = \omega s_1^2 \sin\lambda,$$

and

$$s^2\frac{d\theta}{dt} = \omega \sin\lambda\,(s_1^2 - s^2).$$

If in addition the initial position of the pendulum coincides with the vertical, $s_1 = 0$, and we have for the horizontal angular velocity

$$\frac{d\theta}{dt} = -\,\omega\sin\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The minus sign in equation (7) indicates that the angular velocity is from east through south to west in north latitude, or clockwise to one facing the north.

From (7) we have, if $\theta = 0$ when $t = 0$, for the angle of s with the axis of X

$$\theta = -\,t\omega\sin\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In 24 hours, then, the angle of s with the axis of X, if the pendulum starts from A in the preceding figure and swings initially along AX east and west, will be $2\pi \sin\lambda$. At the pole, then, s will describe a complete circle. At the equator s will not deviate from its initial direction. In latitude 30° north, θ in 24 hours will be π radians or 180° from east to west, or clockwise to one facing the north. In latitude 30° north, θ in 24 hours will be 180° counter-clockwise to one facing the north.

EXAMPLES.

(1) *A locomotive weighing 27 tons runs at the rate of 45 miles per hour on a straight track in latitude 30° north. Find the pressure on the rails (a) when it runs north; (b) south; (c) east; (d) west.* $(g = 32.)$

Ans. (*a*) 290 poundals or about 9 pounds on the east rail; (*b*) the same on the west rail; (*c*) the same on the south rail; (*d*) the same on the north rail.

(2) *In the preceding example let the latitude be 30° south.*

Ans. (*a*) 290 poundals or about 9 pounds on the west rail; (*b*) the same on the east rail; (*c*) the same on the north rail; (*d*) the same on the south rail.

(3) *In example (1) find the vertical pressure on the rails when the locomotive runs (a) north; (b) south · (c) east; (d) west.* $(r = 3960$ *miles.*)

Ans. (*a*) Less by 12.6 poundals or about 0.4 pounds; (*b*) the same; (*c*) less by 515 poundals or about 16 pounds; (*d*) increased by 490.2 poundals or about 15.3 pounds.

(4) *Find the velocity of a body in order that it may have no weight when it moves, in latitude* 60°, (*a*) *north;* (*b*) *south;* (*c*) *east;* (*d*) *west.* ($r = 3960$ *miles*, $g = 32$.)

Ans. (*a*) and (*b*) about 5 miles per sec.; (*c*) about 4.85 miles per sec.; (*d*) about 5.14 miles per sec.

(5) *A particle in latitude* 30° *north has a velocity of* 60 *feet per sec. and moves on a perfectly smooth horizontal plane. Disregarding resistance of the air, find the acceleration and the distance described in latitude and longitude in* 4 *seconds* (*a*) *when the velocity is north;* (*b*) *south;* (*c*) *east;* (*d*) *west.* ($r = 3960$ *miles.*)

Ans. (*a*) $f_x = 0.00436$ ft.-per-sec. per sec. east, distance 240 ft. north and 0.035 ft. east.

(*b*) $f_x = 0.00436$ ft.-per-sec. per sec. west, distance 240 ft. south and 0.015 ft. west.

(*c*) $f_y = 0.05236$ ft.-per-sec. per. sec. south, distance 240 ft. east and 0.42 ft. south.

(*d*) $f_y = 0.05236$ ft.-per-sec. per sec. north, distance 240 ft. west and 0.42 ft. north.

(6) *A cannon-ball is fired in latitude* 30° *north with a velocity of* 1440 *ft. per sec. Neglecting resistance of the air, find the acceleration and distance described in latitude and longitude in* 4 *seconds,* (*a*) *when the velocity is north;* (*b*) *south;* (*c*) *east;* (*d*) *west.*

Ans. (*a*) $f_x = 0.10472$ ft.-per-sec. per sec. east, distance 5760 ft. north and 0.84 ft. east.

(*b*) $f_x = 0.10472$ ft.-per-sec. per sec. west, distance 5760 ft. south and 0.84 ft. west.

(*c*) $f_y = 0.15272$ ft.-per-sec. per sec. south, distance 5760 ft. east and 1.22 ft. south.

(*d*) $f_y = 0.15272$ ft.-per-sec. per sec. north, distance 5760 ft. west and 1.22 ft. north.

(7) *A particle in latitude* 60° *north falls from rest a distance of* 1296 *ft. to the ground. Find the deviation in latitude, disregarding resistance of the air.*

Ans. 0.2827 ft. towards the east.

(8) *In latitude* 30° *north, find the angular velocity of rotation of the plane of a pendulum.*

Ans. 0.0000363 radians per sec. in a direction clockwise to one facing the north. The plane rotates through 180° in 24 hours.

(9) *A locomotive weighing* 32 *tons runs at the rate of* 45 *miles per hour in latitude* 30° *north in a direction S.* 30° *E. on a curve of one mile radius in a counter-clockwise direction to one looking north. Find the pressure on the outer rail.*

Ans. 1868 pounds. If we disregard rotation of the earth, the pressure would be 1848 pounds.

(10) *In the preceding example suppose the direction is clockwise to one looking north.*

Ans. 1825.6 pounds.

CHAPTER III.

IMPULSE. MOMENTUM. STRESS.

IMPULSE. MOMENTUM. NEWTON'S LAWS OF MOTION. STRESS.

Impulse.—Let a uniform force be F acting in any direction and the time of its action be t. Then we call the quantity Ft the impulse of the force in that direction, and denote it by ϕ.

We have then

$$\phi = Ft.$$

The *direction* of the impulse is the same as the direction of the force.

Hence, *impulse is the product of a uniform force by its time of action, and it acts in the direction of the force.*

Line Representative of Impulse.—Impulse then has magnitude and direction, and we can represent it by a straight line like force. The principles, therefore, of pages 70, 84, 95, Vol. I, *Kinematics*, hold good for impulse as well as force, and we can resolve and combine impulses, and have the "triangle and polygon of impulses" as well as of forces.

Unit of Impulse.—If $[F]$ is the unit of force and F the number of units of uniform force, $[T]$ the unit of time and t the number of units of time, $[\phi]$ the unit of impulse and ϕ the number of units of impulse, we have by definition

$$\phi[\phi] = F[F] \times t[T].$$

We have then the numeric equation

$$\phi = Ft,$$

provided that

$$[\phi] = [F] \times [T].$$

The unit of impulse is then the impulse of one unit of uniform force acting for one unit of time.

The English absolute unit of impulse is therefore the *poundal-second*, or the impulse of a uniform force of one poundal acting for one second. We may write it "*pdl.-sec.*"

The C. G. S. absolute unit of impulse is the *dyne-second*, or the impulse of a uniform force of one dyne acting for one second. We may write it "*dyne-sec.*"

In gravitation units we have then the *pound-second* (*lb.-sec.*) or the *gram-second* (*gr.-sec.*).

When, then, we say that "the impulse in any direction on a particle is $\phi = Ft$," we mean that a uniform force F acts in that direction for t seconds, and that this impulse is the same as that of a uniform force of ϕ units acting in that direction for one second.

Momentum.—Let the mass of a particle be m, and its velocity in any direction be v. Then we call the quantity mv the momentum of the particle in that direction, and denote it by n.

We have then

$$n = mv.$$

The *direction* of the momentum is the same as the direction of the velocity.

Hence, *the momentum of a particle in any direction is the product of its mass by its velocity in that direction.*

Line Representative of Momentum.—Momentum then has magnitude and direction, and we can represent it like velocity by a straight line. The principles, then, of pages 70, 84, 95, Vol. I, *Kinematics*, hold good and we can resolve and combine momentums and have the "triangle and polygon of momentum" as well as of velocity and force.

Unit of Momentum.—If $[M]$ is the unit of mass and m the number of units of mass, $[V]$ the unit of velocity and v the number of units of velocity, $[N]$ the unit of momentum and n the number of units of momentum, we have by definition

$$n[N] = m[M] \times v[V].$$

We have then the numeric equation

$$n = mv,$$

provided that

$$[N] = [M] \times [V].$$

The unit of momentum is then the momentum of one unit of mass moving with one unit of velocity. We may call a unit of velocity, or one unit of length per unit of time, a "*velo.*"

The English unit of momentum, then, is the *pound-velo* (*lb.-velo*), or the momentum of a mass of one pound moving with a velocity of one foot per second.

The C. G. S. unit of momentum is then the *gram-velo* (*gr.-velo*) or the *kilogram-velo* (*kil.-velo*), that is, the momentum of a mass of one gram or one kilogram moving with a velocity of one centimetre per sec. A committee of the British Association have proposed for this the name **bole.**

Relation between Impulse and Momentum.—A force, as we have seen (page 2), is *uniform* when it does not change in magnitude or direction. When either the magnitude or the direction changes it is *variable.*

Let a particle of mass m have the initial velocity v_1 in any given direction, and, under the action of a *uniform* force F_u acting in that direction during the time t, acquire the final velocity v in that direction. Then the path of the particle is a straight line, the motion is uniformly accelerated, and (page 51, Vol. I, *Kinematics*) the uniform acceleration is

$$f = \frac{v - v_1}{t}.$$

Hence by the equation of force (page 2)

$$F_u = mf = \frac{m(v - v_1)}{t}, \quad \text{or} \quad F_u t = mv - mv_1. \quad . \quad . \quad . \quad (1)$$

Dividing by t, we obtain

$$F_u = \frac{mv - mv_1}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But we have defined $F_u t$ as the impulse ϕ of a *uniform* force F_u acting for a time t, and the *direction* of the impulse is the direction of this force. Also by definition mv_1 is the initial and mv the final momentum in the direction of v_1 and v. Hence,

For rectilinear motion under the action of uniform force—

1. *The change of momentum in any direction for any time t is equal to the impulse $\phi = F_u t$ of the uniform force F_n acting in that direction for that time.*

2. *The time-rate of change of momentum in any direction for any time t is equal to the uniform force acting in that direction during that time.*

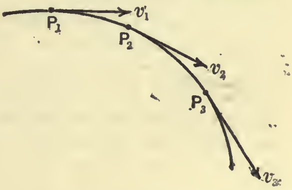
If we take m in pounds and v and v_1 in feet-per-second, equations (1) and (2) give $\phi = F_u t$ in *poundal-seconds* and F_u in *poundals*. For gravitation measure we must divide by g in feet-per-sec. per sec. to obtain *pound-seconds* and *pounds*. So also if we take m in grams and v and v_1 in centimetres per second we obtain ϕ in *dyne-seconds* and F_u in *dynes*. For gravitation units we must divide by g in centimetres-per-sec. per sec. to obtain *gram-seconds* and *grams*.

If ϕ comes out minus, v_1 is greater than v and the impulse and force are opposite to v_1.

A variable force may be considered as uniform during an indefinitely small time.

Let then a particle of mass m be acted upon by a variable force during the time t, and have the initial velocity v_1. The path of the particle is now a curve.

Let P_1, P_2, P_3, etc., be points of the path and v_1, v_2, v_3, etc., the corresponding velocities, and the corresponding times of passing from P_1 to P_2, P_2 to P_3, etc., be t_1, t_2, t_3, etc.



Let the variable force acting upon the particle be resolved into a normal and a tangential component. The normal component causes no change of speed (page 15). Let the tangential component be F_1, F_2, F_3, etc., at the points P_1, P_2, P_3, etc. If the times t_1, t_2, t_3, etc., are taken indefinitely small, then the small arcs P_1P_2, P_2P_3, etc., are practically straight lines, the projection of v_1 upon P_1P_2 is practically equal to v_1, of v_2 upon P_2P_3 practically equal to v_2, and the forces F_1, F_2, etc., in these directions are each practically uniform. We have then, if v is the final velocity,

$$F_1t_1 = m(v_2 - v_1), \quad F_2t_2 = m(v_3 - v_2), \quad F_3t_3 = m(v_4 - v_3), \text{ etc.}, \ldots$$
$$F_nt_n = m(v - v_{n-1}).$$

Summing all these impulses, we have

$$F_1t_1 + F_2t_2 + F_3t_3 + \text{etc.} \ldots + F_nt_n = mv - mv_1,$$

or

$$\Sigma Ft = mv - mv_1.$$

We see, then, that in any case, whether the force is uniform or variable, and whatever the path, *The sum of all the actual impulses along the path is equal to the change of momentum in the path.*

Let the sum of all these impulses be equal to the impulse ϕ of an *equivalent uniform* force F_u acting for the given time $t = t_1 + t_2 + t_3 + \ldots t_n$. Then we have

$$\phi = F_u t = mv - mv_1, \quad \ldots\ldots\ldots \quad (3)$$

and

$$F_u = \frac{mv - mv_1}{t}. \quad \ldots\ldots\ldots \quad (4)$$

Let the forces F_1, F_2, F_3, etc., acting along the path be constant in magnitude and each equal to the tangential force F_t. Then we have

$$F_t(t_1 + t_2 + t_3 + \ldots t_n) = mv - mv_1,$$

or, since t is the entire time,

$$F_t t = mv - mv_1, \quad \ldots\ldots\ldots \quad (5)$$

or

$$F_t = \frac{mv - mv_1}{t}. \quad \ldots\ldots\ldots \quad (6)$$

Hence in all cases, *however the actual force may vary*—

1. *The change of momentum in the path in any time is equal to the sum of all the actual impulses along the path—or is equal to the sum of the impulses along the path of an equivalent tangential force* F_t *of constant magnitude—or is equal to the impulse* $\phi = F_u t$ *of an equivalent uniform force* F_u *acting for that time.*

2. *The time-rate of change of momentum in the path gives either the equivalent uniform force* F_u *or the equivalent tangential force of constant magnitude* F_t, *which, acting for that time, would cause that change of momentum.*

[In calculus notation the rate of change of speed at any point of the path at any instant (page 25, Vol. I, *Kinematics*) is $\frac{dv}{dt}$, and this is the magnitude of the tangential acceleration at that point. We have then for the tangential force at that point

$$F_t = m\frac{dv}{dt}. \quad \ldots\ldots\ldots \quad (1)$$

The *impulse* of this force is

$$F_t dt = m dv. \quad \ldots\ldots\ldots \quad (2)$$

The impulse of an *equivalent uniform* force F_u for the same time is $F_u dt$, and if this impulse is taken equal to the impulse of the tangential force for the same time, we have

$$F_u dt = m dv.$$

Integrating (1) and assuming F_t *constant in magnitude*, we have

$$F_t t = mv + \text{Const.}$$

If $v = v_1$ when $t = 0$, we have Const. $= - mv_1$, and hence for any time t

$$F_t t = mv - mv_1.$$

This is equation (6) just found.

In the same way we find

$$F_u t = mv - mv_1.$$

This is equation (3) just found.

COR. 1. From equation (4) we see that as the time t decreases the uniform force F_u must increase for the same change of momentum. If t is zero, F_u becomes infinitely great. That is, *a given change of momentum requires time*, and the less the time the greater must be the uniform force to produce the change.

COR. 2. If the time is one second and the initial velocity $v_1 = 0$, or the final velocity $v = 0$, we have, from (4) or (6),

$$F_u = \frac{mv}{1 \text{ sec}} = F_t, \quad \text{or} \quad F_u = -\frac{mv_1}{1 \text{ sec}} = F_t.$$

That is, the momentum mv is *numerically equal* to the uniform force F_u or the tangential force F_t of constant magnitude which, acting respectively in the direction of the velocity or along the path in the direction of motion, would give the particle its velocity v starting from rest *in one second*; or which, acting respectively opposite to the velocity or along the path opposite to the motion, would bring the particle to rest *in one second*.

Impulsive Force.—The form of the equation of motion given by equation (4),

$$F_u = \frac{mv - mv_1}{t},$$

is convenient when the magnitude of the force is great and its time of action small, as in cases of impact, collision, etc.

Such forces are therefore called **impulsive forces.** As we have seen, however, equation (4) holds whatever the time, and hence the restriction of the term "impulsive force" to one whose time of action is very short is simply a matter of convenience.*

Newton's Laws of Motion.—In 1687 the facts of motion of material particles were stated by Newton in the form of three laws known as **Newton's Laws of Motion.** Simple as these laws appear, the science of Dynamics made no essential progress until they were recognized.

These laws are statements of facts of nature, not *a priori* deductions, and their proof is found in the accord of the results deduced from them with observed phenomena. The proof thus furnished in Dynamics and Astronomy is of such a nature that these laws are regarded as rigorously true, and deductions made from them are accepted even when such deductions cannot be tested directly by experiment.

The two centuries which have elapsed since the statement of these laws by Newton have not made necessary any additions or modifications except in the terms employed.

Newton's First Law of Motion. — This law was expressed by Newton as follows:

LEX I. *Corpus omne perseverare in statu suo quiescendi vel*

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*The term "impulse" is unfortunately applied by some writers to a short-lived force itself, instead of the term *impulsive force* as used above. This makes it necessary to speak of the "impulse of an impulse" when we wish to speak of *impulse* as used above. The term "impulsive force" has also been used to denote the *impulse* of a short-lived force as used above, thus leaving no term to denote the short-lived force itself.

movendi uniformiter in directum, nisi quatenus illud à viribus impressis cogitur statum suum mutare.

Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled to change that state by impressed forces.

This first law asserts, then, the property of inertia for all bodies. It also implicitly defines force as that which causes a material particle to change its motion either in direction or speed. Thus it states that no body can change the direction or speed of its own motion of rectilinear translation. This is the property of inertia (page 1, Vol. II, *Statics*). It also states that such a change is due to "impressed," that is *external*, force, or the influence of other bodies.

Whenever, then, we find all the particles of a body moving with uniform speed in a straight line, we know that either it is not acted upon by external forces, or else these forces must mutually balance and the body moves as if they did not exist.

Whenever the speed of a particle changes or the direction of motion changes, we know that some external influence or force causes the change.

Taking velocity, then, as defined (page 42, Vol. I, *Kinematics*), the law states that change of velocity of a particle is due to force and this force is due to the influence of external bodies. Without such influence the velocity is uniform. Force, then, is proportional to acceleration, using the term acceleration as defined on page 48, Vol. I, *Kinematics*.

The law is in direct contradiction to the tenets of the ancient philosophers, who maintained that circular motion was "perfect" and "natural." There can be no circular motion without force.

Newton's Second Law of Motion.—The phrase "except in so far" prepares the way for the statement of the second law :

LEX II. *Mutationem motûs proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam quâ vis illa imprimitur.*

Change of motion is proportional to the motive force, and takes place in the direction of the straight line in which the force acts.

By "motion" Newton here refers, not to velocity as we have defined it, but to *mass-motion* or what we have designated as *momentum*, and his "change of motion" is what we have called *impulse*. This law, then, is the statement of the equation $F_u t = m(v - v_1)$, and asserts that the change of momentum is proportional to the force which produces it and is in the same direction. This second law tells us then how to measure force when it exists.

Newton's Third Law of Motion.—When one body presses against or pulls another, it is itself pressed or pulled by this other with an equal force in the opposite direction.

When one body has its momentum changed in direction or amount by impact, the other body has its momentum changed by the same amount in the opposite direction. For at each instant during impact the forces between them are equal and opposite. When one body attracts another, this other always attracts it with equal and opposite force.

Taking into account, then, the entire phenomenon of the mutual action and reaction between two portions of matter, we have the third law of Newton :

LEX III. *Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuò semper esse æquales et in partes contrarias dirigi.*

To every action there is always an equal and contrary reaction:

or the mutual actions of any two bodies are always equal and oppositely directed.

Stress.—The exertion of force upon a body is thus only one side of the entire phenomenon, which is the simultaneous exertion of equal and opposite forces between two bodies.

When we fix our attention upon one only of these bodies and, disregarding the other, consider only its action upon the first, we call this action *force.* But when we have both bodies in mind and wish to be understood as viewing this force as one of the two mutual, equal and opposite actions between two bodies or between two parts of the same body, we call it a **stress.**

When the stress is such as to make the bodies move towards one another or to resist extension it is attraction or **tensile stress.** When its effect is to increase their distance or to resist compression it is repulsion or **compressive stress.**

In this sense we always speak of the stress *in* a body or the stress *between* two bodies, the prepositions "in" and "between" indicating at once that one of the mutual actions between two bodies or parts of the same body is meant.

Stress as thus used is then always internal, while force is always external, to the body or system under consideration. (See page 7, Vol. II, *Statics.*)

External Stress.—There is a sense, however, in which we may speak of stress *on* a body and thus consider it as external, which need never be confounded with that just given.

Force is often exerted upon some portion of the bounding surface of a body and acts then over an area. In such case the number of units in its magnitude divided by the number of units in this area gives the number of units of *force per unit of area.*

When a force thus acts we may speak of it as the stress *on* the body, and of the force per unit of area as the **unit stress.**

This use of the word stress is convenient and leads to no confusion. When necessary to discriminate we may call it **external stress**, but in general such distinction is unnecessary, as the use of the prepositions "on" and "in" sufficiently indicate the sense in which the term is used.

EXAMPLES.

[See page 97, Vol. I, for equations of motion of a point.]

(1) *A ball-player catches a ball moving with a velocity of* 50 *ft. per sec. The mass of the ball is* 5½ *oz. If the space in which the ball is brought to rest is* 6 *inches, what is the pressure on the hands, supposed uniform? What is the time of stoppage?*

Ans. We have

$$s = \frac{vt}{2}, \quad \text{or} \quad t = \frac{2 \times \frac{1}{2}}{50} = \frac{1}{50} \text{ sec.}$$

The pressure is $F = \frac{mv}{t} = \frac{11 \times 50 \times 50}{32} = 859\frac{3}{8}$ poundals or about 26.85 lbs., taking $g = 32$ ft.-per-sec. per sec.

(2) *An* 80-*ton gun on a smooth horizontal plane fires horizontally a shot of* 56 *lbs. with a velocity V of* 1800 *ft. per sec. Find the velocity of recoil, v.*

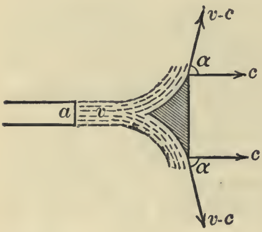
Ans. Mass of gun $= 80 \times 2240 = 179200$ lbs. If the velocity is imparted in the time t, the uniform force which would give that velocity in that time, starting from rest, to the shot is $\frac{mV}{t} = \frac{56 \times 1800}{t} = \frac{100800}{t}$ poundals. Since action and reaction are equal, we have $\frac{179200v}{t} = \frac{100800}{t}$, or $v = \frac{9}{16}$ ft. per sec.

(3) *A man whose weight is* 150 *lbs., moving on a bicycle with a speed of 22 ft. per sec. and wishing to right his bicycle, which is leaning to one side, turns the wheel so that in* 1/10 *sec. his direction of motion is changed* 30°. *What is the force which acts to right the bicycle?*

Ans. The rate of change of momentum in the original direction of motion is $\frac{m(v_1 - v)}{t} = \frac{150(22 - 22\cos 30°)}{\frac{1}{10}} = 16500$ poundals or 515⅝ pounds, if $g = 32$ ft.-per-sec. per sec. This force in the original direction of motion changes the velocity of the mass in that direction, and since action and reaction are equal this force acting in the original direction of motion on the wheel acts to right it.

(4) *A stream of water whose cross-section is a and velocity v meets a surface moving in the same direction with a velocity c. Disregarding friction, what is the pressure exerted on the surface by the stream?*

Ans. Let the water pass off the surface in a direction making an angle α with the direction of motion. The volume of water in any time t is avt. If γ is the density or mass of a unit of volume of water, the mass in this time is $\gamma avt = m$.



The velocity relative to the surface just before impact, in the direction of motion, is $v - c$. After impact the velocity relative to the surface in the direction of motion is $(v - c)\cos\alpha$. We have then the pressure F equal to the rate of change of momentum in the direction of motion, or

$$F = \frac{m(v - v_1)}{t} = \gamma av(v - c)(1 - \cos\alpha),$$

or in gravitation units

$$F = \frac{\gamma av}{g}(v - c)(1 - \cos\alpha).$$

(5) *A mass of* 10 *lbs. under the action of a uniform force receives in* 3 *seconds an integral acceleration in the direction of the force of* 6 *ft. per second. What is the force?*

Ans. The acceleration is 2 ft.-per-sec. per sec. The force is $10 \times 2 = 20$ poundals, or the weight of $\frac{20}{g}$ lbs.

(6) *A uniform force of* 200 *dynes changes the velocity of a body moving in a straight line from* 250 *to* 300 *metres per sec. in* 1 *minute. Find the mass of the body.*

Ans. 2.4 grams.

(7) *A mass of* 10 *lbs. moving in a straight line with a velocity of* 3 *ft. per sec. is brought to rest by a uniform opposing force in* 2 *sec. What is the force?*

Ans. 15 poundals, or the force of gravity upon $\frac{15}{g}$ lbs.

(8) *A rifle-bullet weighing one ounce is shot into a block of wood weighing* 53 *pounds, and gives the block a velocity of* 2 *ft. per sec. in* 1 *sec. What was the velocity of the bullet?*

Ans. 1698 ft. per sec. The mass of the bullet should be added to that of the block.

(9) *A ship weighing* 336,000 *lbs. runs upon a rock with a velocity of* 16 *miles an hour. Assuming the force of stoppage as uniform and the time of stoppage* 2 *sec., what is this force?*

Ans. 3942400 poundals, or the force of gravity upon $\frac{3942400}{g}$ lbs. Taking $g = 32$ ft.-per-sec. per sec., 123200 pounds.

(10) *A mass moving in a straight line with a velocity of* 3 *ft. per sec. is brought to rest by a uniform opposing force of one pound in* 2 *sec. Assuming* $g = 32$ *ft.-per-sec. per sec., what is the mass?*

Ans. $21\frac{1}{3}$ pounds.

(11) *A uniform force of* 10 *lbs. acts for* 2 *sec. upon a mass of* 10 *lbs. and then ceases. With what velocity will the mass continue to move in the direction of the force?*

Ans. $2g$ ft. per sec.

(12) *A mass of* 20 *lbs. moving with a velocity of* 15 *ft. per sec. in a straight line is found after* 3 *sec. to be moving in the same direction with a velocity of* 5 *ft. per sec. What is the retarding force, assuming it uniform?*

Ans. $66\frac{2}{3}$ poundals, or the force of gravity upon $\frac{66\frac{2}{3}}{g}$ pounds.

(13) *A baseball weighing* $5\frac{1}{2}$ *oz. is dropped from the top of a tower* 1000 *ft. high. What uniform pressure must a catcher apply to it in order to bring it to rest in* 5 *feet?*

Ans. 68.75 lbs. if $g = 32$ ft.-per-sec. per sec.

(14) *How long must a uniform force of* 14 *lbs. act on a mass of* 1000 *tons* (2240 *lbs.*) *to give it a velocity of one foot per second?* ($g = 32$ *ft.-per-sec. per sec.*)

Ans. 5000 sec.

(15) *Calculate in pounds the uniform moving force which, acting for a minute upon the mass of a ton* (2240 *lbs.*), *will get up in it a velocity of* 30 *miles an hour.* ($g = 32$ *ft.-per-sec. per sec.*)

Ans. 50 pounds.

(16) *A body of* 3 *lbs. mass is falling at the rate of* 100 *ft. per sec. Find the uniform force that will stop it in* 2 *seconds; in* 2 *feet.* ($g = 32$ *ft.-per-sec. per sec.*)

Ans. $4\frac{11}{16}$ lbs.; $234\frac{3}{8}$ lbs.

(17) *A cannon-ball of* 1000 *grams mass is discharged with a velocity of* 45000 *centimetres per sec. from a cannon the length of whose barrel is* 200 *centimetres; show that the mean force exerted on the ball is* 5.0625×10^9 *dynes.*

(18) *It was found that when* 1 *foot was cut off from the barrel of a gun firing a projectile of* 100 *lbs. the velocity was changed from* 1490 *to* 1330 *ft. per sec. Show that the total pressure at the muzzle was about* 315 *tons* (2240 *lbs.*). ($g = 32$ *ft.-per-sec. per sec.*)

(19) *A particle of* 10 *lbs. mass has an initial velocity of* 20 *ft. per sec. towards the north and is acted upon by two uniform forces,*

one of 100 *poundals in a direction northeast, and the other of the same magnitude in a direction northwest. Find its velocity after* 1 *minute.*

Ans. The resultant force is 141.41 poundals acting towards the north. We have then (page 34) $Ft = m(v - v_1)$, or $141.41 \times 60 = 10(v - 20)$. Hence $v = 868.5$ ft. per sec. towards the north.

(20) *A particle of mass m is moving towards the east with a velocity v. Find the uniform force necessary to make it move towards the north with an equal velocity in t seconds.*

Ans. In the time t the velocity towards the east becomes zero. The impulse of the constant force *towards the west* must then be (page 32) mv. The impulse of the constant force towards the north is also mv. The impulse of the resultant force is then $Ft = mv\sqrt{2}$. Hence $F = \frac{mv\sqrt{2}}{t}$ towards the northwest.

(21) *A uniform force of* 20 *poundals acts for* 5 *secs. on a particle of mass* 10 *lbs. The initial velocity is* 4 *ft. per sec., making an angle of* 60° *with the direction of the force. Find the velocity at the end of the time.*

Ans. The force in the direction of the initial velocity is $20 \cos 60° = 10$ poundals, and at right angles $20 \sin 60° = 20\sqrt{\frac{3}{4}} = 10\sqrt{3}$ poundals. Since $Ft = m(v - v_1)$ in the direction of the force (page 34), we have for the velocity in the direction of the force

$$10 \times 5 = 10(v - 4), \quad \text{or} \quad v = 9 \text{ ft. per sec.},$$

and for the velocity at right angles $10\sqrt{3} \times 5 = 10v$, or $v = 5\sqrt{3}$ ft. per sec.

The resultant velocity is then $\sqrt{156} = 2\sqrt{39}$ ft. per sec., and it makes with the initial velocity an angle whose sine is $\frac{5}{2\sqrt{13}}$.

CHAPTER IV.

WORK. POWER.

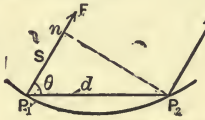
WORK. UNIT OF WORK. WORK OF THE RESULTANT. WORK AND MOMENTUM. WORK OF A TANGENTIAL FORCE OF CONSTANT MAGNITUDE. WORK OF VARIABLE FORCE IN GENERAL. WORK UNDER GIVEN FORCES. POWER. UNIT OF POWER. EFFICIENCY. MECHANICAL ADVANTAGE.

Work.—A force, as we have seen (page 2), is uniform when it does not change in magnitude or direction. When either the magnitude or direction changes it is variable.

When a *uniform* force F acts upon a particle in any given direction and the displacement of the particle *along the line of the force* is s, the product Fs is called **work**. If the displacement s is in the same direction as the force, work is said to be done by the force. If the displacement s is opposite in direction to the force, work is said to be done *against* the force. In the first case the work is positive (+), in the second case negative (−). In both cases the magnitude of the force is given by Fs.

Thus let a uniform force F act upon a particle P_1, and let the *displacement* (page 34, Vol. I, *Kinematics*) during the action of the force, *whatever the path of the particle may be*, be $P_1P_2 = d$.

Let this displacement make the angle θ with the uniform force. Then the displacement *along the line of the force* is



$$P_1n = s = d\cos\theta.$$

The work of F is then by definition

$$W = \pm Fs = \pm Fd\cos\theta,$$

the (+) sign being used when the displacement is in the direction of the force, as in the figure, and the (−) sign when it is opposite to that direction.

But we see that $F\cos\theta$ is the component of the force F *along the line of the displacement.*

We can then define work generally as follows:

Work is the product of a uniform force by the component, along the line of that force, of the displacement of the particle on which the force acts; or, the product of the displacement by the component, along the line of the displacement, of a uniform force.

Cor. 1. We see at once that *work is independent of time.* A given uniform force and displacement give the same work no matter whether the time in which the displacement takes place is large or small. It is also *independent of the path.* For the same uniform force and the same displacement the work is the same whatever the path of the particle.

Cor. 2. The work done in raising a body is equal to the weight of the body which acts at the centre of mass (page 18, Vol. II, *Statics*) multiplied by the vertical displacement of the centre of mass. This work is done *against* the weight and is therefore negative.

Also, the work of lowering a body is equal to its weight multiplied by the vertical displacement of its centre of mass. This work is done *by* the weight and is therefore positive.

If m is the mass, then mg is the weight. If s is the vertical displacement, then the work in general is

$$W = \pm mgs.$$

For the same weight and the same vertical displacement this work is the same, whatever the time of displacement and whatever the path of the centre of mass.

Units of Work.—If $[F]$ is the unit of force and F the number of units, $[L]$ the unit of distance and s the number of units of displacement in the direction of the force, we have

$$W[W] = F[F] \times s[L],$$

or

$$W = Fs \quad \text{if} \quad [W] = [F][L].$$

The unit of work is then the unit of force acting through the unit of distance.

The English absolute unit of work is thus the *foot-poundal*, or a constant force of one poundal acting through one foot.

The C. G. S. absolute unit of work is a constant force of one dyne acting through one centimetre. It is called an **erg**.

A multiple of the latter equal to 10,000,000 ergs, or 10^7 ergs, is used in electrical measurements and called a *joule*, after Dr. James Prescott Joule.

In English gravitation units we have the *foot-pound*. This is the unit commonly adopted in engineering calculations. It is the work of raising one pound through the vertical distance of one foot against the force of gravity. It is then a variable amount of work, since the weight of one pound varies at different localities.

If in the corollary of the preceding article we take s in feet and m in pounds, we obtain work in *foot-poundals*. To reduce to foot-pounds we must divide by g.

For work in gravitation units, or foot-pounds, we have then

$$W = ms,$$

where m is the mass in pounds and s the vertical displacement in feet.

Work of the Resultant Equal to the Work of the Components.—Let F_1, F_2, F_3, Fig. 1, be any number of forces in a plane acting upon a point P which undergoes the displacement D in the direction PD.

Let OF_1, F_1F_2, F_2F_3, Fig. 2, be their line representatives. Since forces can be combined like displacements, the resultant is given in magnitude and direction by $OF_3 = R$ (page 59, Vol. II, *Statics*).

FIG. 1.

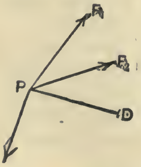
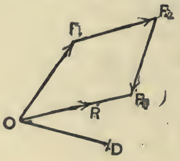


FIG. 2.



Draw OD parallel to PD, and let α, β, γ, θ be the inclinations of F_1, F_2, F_3 and R with the direction of the displacement OD.

Then we have

$$R\cos\theta = F_1\cos\alpha + F_2\cos\beta - F_3\cos\gamma.$$

That is, the component of the resultant R in the direction of the displacement is equal to the algebraic sum of the components of the forces in this direction.

Multiplying by the displacement D, we have

$$R \times D\cos\theta = F_1 \times D\cos\alpha + F_2 \times D\cos\beta - F_3 \times D\cos\gamma.$$

Hence, *the work of the resultant of any number of forces in a plane acting on a point is equal to the algebraic sum of the works of the components.*

The same evidently holds true when the forces are not in a plane.

COR. Any number of forces acting on a point are in *equilibrium* when the resultant is zero. In such case we have a system of balanced forces and the motion of the point is not affected by their action.

We have then for equilibrium

$$0 = F_1 \times D\cos\alpha + F_2 \times D\cos\beta - F_3 \times D\cos\gamma.$$

Hence, *when the algebraic sum of the works of any number of forces in a plane, acting on a point, is zero, the forces are in equilibrium, the resultant is zero, and the motion of the point is unaffected by these forces.*

The same holds true when the forces are not in a plane. This is the *principle of virtual work*, page 621, Vol. II, *Statics*.

Relation between Work and Momentum.—If the uniform acceleration in the direction of the uniform force F is f, and v_1 and v are the initial and final velocities *in the direction of the uniform force* during its time of action t, then

$$f = \frac{v - v_1}{t}.$$

Hence by the equation of force (page 2) the uniform force F is

$$F = mf = \frac{mv - mv_1}{t}.$$

If s is the displacement *in the direction of the force*, we have for the work W

$$W = Fs = \frac{mv - mv_1}{t} . s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the same way if v_1 and v are the initial and final velocities *in the direction of the displacement* d, and F is the component of the uniform force *in the direction of the displacement*, then

$$W = Fd = \frac{mv - mv_1}{t} . d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, *work is equal to the time-rate of change of momentum in the direction of the uniform force multiplied by the component displacement in that direction, or to the time-rate of change of momentum in the direction of the displacement multiplied by the displacement.*

If we take s or d in feet and m in pounds, and v_1, v in feet per second, we obtain the work in foot-poundals. For work in gravitation units we must divide by g in feet-per-sec. per sec. We then obtain work in foot-pounds.

If we take s or d in centimetres and m in grams, and v_1, v in centimetres per second, we obtain the work in ergs. For work in gravitation units we must divide by g in centimetres-per-sec. per sec. We thus obtain work in centimetre-grams.

COR. Since v_1 and v are the initial and final velocities in the direction of the uniform force or of the displacement, the mean speed is $\frac{v_1 + v}{2}$, and the distances s or d passed over in the time t are then

$$s = \frac{v_1 + v}{2} . t, \quad \text{or} \quad d = \frac{v_1 + v}{2} . t.$$

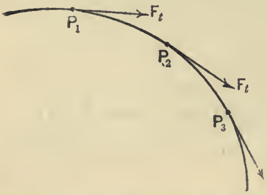
If we substitute these values of s and d in (1) and (2), we obtain in either case

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We see then again that the work is independent of the time t and of the path, and depends simply upon the initial and final velocities v_1 and v in the direction of the uniform force or in the direction of the displacement. (See Cor. 1, p. 41.)

Equation (3) then gives the work in either direction if v_1 and v are the velocities in that direction. (For another demonstration see page 45.)

Work of a Tangential Force of Constant Magnitude.—Let P_1, P_2, P_3, etc., be points of the path of a moving particle, and let the force acting upon the particle be always tangential to the path and constant in magnitude. Denote this force by F_t. This force may be considered as *uniform* for an indefinitely short time. In this indefinitely short time the small arcs described, P_1P_2, P_2P_3, etc., are practically straight lines. Denote their lengths by s_1, s_2, etc. The component of F_t along P_1P_2 is practically equal to F_t, along P_2P_3 the same, and so on. We have then for the total work of F_t in the path



$$W = F_ts_1 + F_ts_2 + F_ts_3 + \text{etc.} = F_t\Sigma s.$$

But Σs is equal to the entire length of path s. Therefore

$$W = F_ts. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, *the work of a tangential force of constant magnitude is equal to the product of the force by the length of path.*

If v_1 is the initial and v the final velocity in the path, and t the time of describing the path, then, as we have seen (page 34).

$$\frac{mv - mv_1}{t}$$

is the tangential force F_t of constant magnitude which would give the particle of mass m the change of speed $v - v_1$. We have then from (1)

$$W = \frac{mv - mv_1}{t} . s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For work in gravitation units we must divide by g.

Hence, *the work of a tangential force of constant magnitude is equal to the time-rate of change of momentum in the path multiplied by the length of path described.*

COR. 1. Since the mean speed is $\frac{v + v_1}{2}$, the distance s is $\frac{v + v_1}{2} . t$, and we have from (2)

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here we see again that the work is independent of the time and path and depends simply upon the initial and final velocities v_1 and v. (See Cor., page 44, Cor. 1, page 41; also page 44 for another demonstration.)

COR. 2. If the path is a circle of radius r, then $v = r\omega$, where ω is the angular velocity, and we have

$$W = \frac{1}{2}mr^2\omega^2 - \frac{1}{2}mr^2\omega_1^2.$$

Work of a Variable Force in General.—Let a particle of mass m move in any path under the action of any number of variable forces during the time t. The forces acting upon the particle at any point of its path can be resolved into a resultant normal and tangential component. The normal component does no work. Since the work of the resultant is equal to the algebraic sum of the works of the components, the work of the variable forces is equal to the work of the resultant variable tangential component.

Let v_1 be the initial and v the final velocity. Then, as we have seen (page 34), the tangential force of *constant magnitude* which would cause the given change of momentum $mv - mv_1$ in the path is

$$F_t = \frac{mv - mv_1}{t}.$$

The mean speed in the path is $\frac{v + v_1}{2}$. Hence the length of path is

$$s = \frac{v + v_1}{2} . t.$$

Therefore from (1), page 44, the work done whatever the number of forces, whatever the time or path, whether the forces are uniform or variable, is

$$W = F_t s = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We shall discuss this result more at length under the head of *energy* in the next chapter. (See page 44 for another demonstration.) It is sufficient to call attention here to the fact that this result is general and includes all cases. (See Cor., page 44, Cor. 1, page 41, Cor. 2, page 45.)

[Equation (1) is easily deduced by the Calculus as follows. The acceleration at any instant is

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The force at that instant is then

$$mf = m\frac{d^2s}{dt^2}.$$

The differential of the work is then

$$dW = m\frac{d^2s}{dt^2}ds = m\frac{ds}{dt}.d\left(\frac{ds}{dt}\right).$$

Integrating,

$$W = \frac{1}{2}m\frac{ds^2}{dt^2} + \text{Const.},$$

or, since $\frac{ds}{dt} = v$,

$$W = \frac{1}{2}mv^2 + \text{Const.}$$

When $v = v_1$ let $W = 0$. Then Const. $= -\frac{1}{2}mv_1^2$ and we have

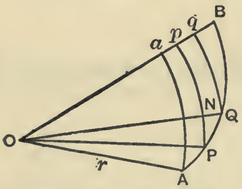
$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2.$$

Cor. If the path is a circle of radius r, we have $v = r\omega$, where ω is the angular velocity. In this case we have

$$W = \frac{1}{2}mr^2\omega^2 - \frac{1}{2}mr^2\omega_1^2.$$

Work Done under Given Forces.—(*a*) **Uniform Force.**—When a particle is acted upon by a uniform force, the work done is the product of the force by the component displacement in its direction, or the product of the displacement by the component force in its direction.

(*b*) **Central Force.**—Let O be the centre of force, the force being attractive or always towards the centre O. Let BA be any path of a particle from B to A. Take any indefinitely small portion of the path QP, so that the force between Q and P may be considered constant and equal to F, its direction being QO.



With O as a centre draw arcs of circles through Q, P and A, intersecting BO at q, p and a. Join BO, QO, PO, AO.

Then QPN is a triangle right-angled at N, and the work done in moving the particle from Q to P is

$$F \times QP \cos NQP, \quad \text{or} \quad F \times QN = F \times qp.$$

Every element of the path may be treated in the same way. Therefore the work necessary to move the particle from B to A, by *any path*, under the action of a central force always directed towards O, is equal to that necessary to move it from B to a in the straight line BO. This work is then *independent of the path*, and depends only on the final and initial positions and the magnitude of the force.

If the magnitude of the force is constant and equal to F, the work is $F(R - r)$, where $R = BO$ and $r = AO$.

(*c*) **Central Force Proportional to Distance from the Centre.** — If the magnitude of the force varies directly as the distance from the centre, let F' be its magnitude at any given distance r'. Then the force F at A is given by $F : F' :: r : r'$, or $F = F'\frac{r}{r'}$, and at B, $F = \frac{F'R}{r'}$. The mean force is $\frac{F'}{2r'}(R + r)$, and the work is

$$\frac{F'}{2r'}(R + r)(R - r), \quad \text{or} \quad W = \frac{F'}{2r'}(R^2 - r^2),$$

where $R = BO$ and $r = AO$.

(*d*) **Central Force Inversely Proportional to the Square of the Distance from the Centre.**—Let as before F' be the force at a distance r'. Then the force at P is $\frac{F'r'^2}{OP^2}$, and at Q it is $\frac{F'r'^2}{OQ^2}$. Since QP is indefinitely small, we can write for both of these $\frac{F'r'^2}{OP \times OQ}$. The work in passing from Q to P is then $\frac{F'r'^2}{OP \times OQ}(QO - PO)$, or $F'r'^2\left(\frac{1}{OP} - \frac{1}{OQ}\right)$. In the same way we have for the work from P to A, $F'r'^2\left(\frac{1}{OA} - \frac{1}{OP}\right)$, and for that from B to Q, $F'r'^2\left(\frac{1}{OQ} - \frac{1}{OP}\right)$. Adding these, we have for the total work from B to A

$$W = F'r'^2\left(\frac{1}{r} - \frac{1}{R}\right),$$

where $R = BO$ and $r = AO$.

COR. Hence the work in passing from an infinite distance $R = \infty$ to a distance $r = AO$ is $W = \frac{F'r'^2}{r}$.

EXAMPLES.

(1) *A body of* 80 *pounds is projected along a rough horizontal plane with a speed of* 50 *ft. per sec. If the constant retarding force of friction is equal to* 20 *lbs., find the work done against friction in the first second; the total work done in coming to rest.* ($g = 32.2$.)

Ans. 919.5 ft.-lbs.; 3105.6 ft.-lbs.

(2) *Show that the work done in drawing a heavy body up a rough inclined plane is the same as if the body were drawn along the equally rough base and then lifted through the vertical height.*

(3) *The distance between two places is* 105 *miles. Train A stops at* 27 *stations. Train B runs through without stopping. The average resistances to A and B with the brakes off are equal to* 1/280 *and* 1/224 *of their respective weights. With the brakes on, the resistances are in both cases* 1/28 *of the respective weights. Suppose the brakes to be always applied when the speed has been reduced to* 30 *miles per hour and not before. Find which train is more expensive, and by how much per cent.*

Ans. Train A, by 9.4 per cent.

(4) *Show that, in the case of a particle which is oscillating with a simple harmonic motion, the work done during its motion from its extreme position to its mean position is twice that done during its motion from a distance equal to three fourths of its amplitude to a distance equal to one fourth of its amplitude.*

(5) *Find the work done by the sun's attraction during the motion of the earth from Aphelion to Perihelion.* [*Mass of earth* $= 6.14 \times 10^{27}$ *gms.; mass of sun* $= 327000$ *times that of the earth; distance at Aphelion* $= 1.512 \times 10^{13}$ *cm.; distance at Perihelion* $= 1.462 \times 10^{13}$ *cm.; radius of earth* $= 6.37 \times 10^{8}$ *cm.;* $g = 981$ *cm.-per-sec. per sec.; force at a unit's distance (page* 48, *Vol. II, Statics),* $(M + m_1)gr_1^2$, *where* M *is mass of sun,* m *mass of earth,* r_1 *radius of earth.*]

Ans. 1.79×10^{39} ergs.

(6) *At the three corners* A, B, C *of a square* $ABCD$ *(side* $= 100$ *metres) are material particles of* 3928, 7856 *and* 11784 *grams. Find the work done against gravitational attraction in moving* 1 *gram from the centre to the fourth corner.* [*Force at a unit's distance (page* 48, *Vol. II, Statics),* $(M + 1)\dfrac{gr_1^2}{m_1}$, *where* m_1 *and* r_1 *are the mass and radius of the earth, and* M *the mass of the particle at each corner,* $g = 981$.]

Ans. 7.82×10^{-8} ergs, approximately.

(7) *A train of* 120 *tons* (2240 *lbs.) runs on a level road, and the resistances average* 8 *lbs. per ton. Find the work in a run of* 40 *miles.* ($g = 32.2$.)

Ans. 2.03×10^{8} ft.-lbs.

(8) *The area of the piston of a steam-engine is* A, *the length of stroke* L, *the steam-pressure per unit of area* P, *the number of strokes per minute* N. *Find the work per minute.*

Ans. $P.L.A.N.$

(9) *Find the work per stroke of an engine when the average pressure of steam is* 38 *lbs. per square inch of piston area, the length of stroke* 3 *feet, and the diameter of piston* 14 *inches.*

Ans. 17556 ft.-lbs. per stroke.

(10) *It is found, neglecting friction, that a horizontal force will move* 10 *lbs. up* 5 *feet of incline rising* 1 *in* 4. *Find the work done and the force parallel to the plane which will just support the weight of* 10 *lbs.*

Ans. 12.5 foot-pounds; 2.5 lbs.

(11) *Find the work done, neglecting friction, in drawing a car of* 2 *tons weight, loaded with* 30 *passengers averaging* 154 *pounds each up a slope the ends of which differ in level by* 50 *feet.*

Ans. 4.55×10^{5} ft.-lbs.

(12) *The grade of a mountain path is* 30°. *How much work against gravity is done by a man of* 168 *lbs. in walking a mile?*

Ans. 4.44×10^{5} ft.-lbs.

(13) *Determine the unit of mass in order that the absolute unit of work may be the foot-pound, taking the second and foot as units and* 32.2 *ft.-per-sec. the acceleration due to gravity.*

Ans. 32.2 lbs.

(14) *A hole is punched through a plate of wrought iron one half*

inch thick, the pressure on the punch being 36 *tons. Assuming the resistance to the punch uniform, find the work.*

Ans. 3360 ft.-lbs.

(15) *Find the work done by a crane in lifting the material for a stone wall* 100 *feet long,* 36 *feet high and* 2 *ft. thick, the density of the stone being* 153 *pounds per cubic foot.*

Ans. 1.983×10^7 ft.-lbs.

(16) *A fly-wheel weighing* 7 *tons* (2240 *lbs.*) *turns on a horizontal axle* 1 *foot in diameter. If the resistance of friction is* 3/40 *of the weight, what is the work done in* 10 *turns in overcoming friction?*

Ans. 36,945 ft.-lbs.

(17) *The resistance of friction along an inclined plane is taken at* 150 *lbs. for each ton of weight. Find the work in drawing* 2 *tons* (2240 *lbs.*) *up* 100 *ft. of an incline which rises* 1 *ft. for* 25 *ft. in length.*

Ans. 47,920 ft.-lbs.

(18) *Weights of* 10 *lbs. and* 8 *lbs. are connected by a string which passes over a pulley. It is found that the heavier weight is just less than necessary to move the smaller. If now the weights are moved uniformly through* 12 *ft., find the work done against friction.*

Ans. 24 ft.-lbs.

(19) *The plunger of a force-pump is* $8\frac{3}{4}$ *inches diameter, the length of stroke is* 2 *ft.* 6 *in., and the pressure is* 50 *lbs. per square inch. Find the work per stroke.*

Ans. 7516 ft.-lbs. per stroke.

(20) *Find the equivalent of one foot-poundal in ergs.*

Ans. 421390 ergs.

(21) *Find the multiplier by which ergs are reduced to foot-pounds.*

Ans. 7.37×10^{-8}.

(22) *A particle of mass m moves horizontally in a circular path of radius r ft.* (*a*) *with uniform speed,* (*b*) *with uniform rate of change of speed a. Find the work done in both cases during the motion of the particle through a semicircle.*

Ans. (*a*) none; (*b*) $\pi r^2 ma$ ft.-poundals.

(23) *In the preceding example let the plane of the path be vertical, and the particle move from top to bottom through a semicircle.*

Ans. (*a*) $2mgr$ ft.-poundals; (*b*) $2mgr + \pi r^2 ma$ ft.-poundals.

(24) *If the particle move from right to left, in vertical plane, through a semicircle.*

Ans. (*a*) none; (*b*) $\pi r^2 ma$ ft.-poundals.

Rate of Work—Power.—Work, as we have seen, is independent of time. If now we take time into consideration, the *time-rate of work*, or work per second, is called **Power**.

The mean rate at which a force does work in a given time is the quotient of the work divided by the time. If the mean rate does not vary, it is uniform. If it does vary with the time, it is variable.

The instantaneous rate is the mean rate when the interval of time is indefinitely small.

We have then

$$R = \frac{W}{t} \quad \text{or} \quad R = \frac{dW}{dt}.$$

Unit of Rate of Work.—If $[W]$ is the unit of work, $[F]$ the unit of force, $[L]$ the unit of distance, and $[T]$ the unit of time, we have for the rate of work

$$R[R] = \frac{F[F]s[L]}{t[T]}.$$

We shall have

$$R = \frac{Fs}{t} = \frac{W}{t}$$

if we take

$$[R] = \frac{[F][L]}{[T]} = \frac{[W]}{[T]}.$$

The unit of power is then one unit of work per unit of time. The English absolute unit of power is then 1 ft.-poundal per sec., and the C. G. S. absolute unit is one erg per sec.

A multiple of this equal to 10^7 ergs per sec. is used in electrical measurements and called the *Watt*, after James Watt. The watt is therefore one joule per sec. (page 42).

In gravitation units we have, in English measures, the foot-pound per sec. The unit employed in engineering calculations is 550 ft.-lbs. per sec. or 33000 ft.-lbs. per minute. This is called a **Horse-power** and denoted by H. P.

In French gravitation units we have the metre-kilogram per sec., and in engineering calculations the unit is 75 metre-kilograms per sec., equivalent to 542.486 ft.-lbs. per sec., which is called the **force de cheval.**

Power and Momentum.—We have then

$$R = \frac{Fs}{t} = Fv,$$

where R is the rate of work of the constant force, F and s the displacement, and v the velocity in the direction of the force.

If m is the mass of a particle and f the acceleration due to the force, we have

$$R = mfv, \quad \text{or} \quad f = \frac{R}{mv}.$$

Hence, *the acceleration due to a constant force whose rate of work is R is the quotient of the rate of work divided by the momentum in the direction of the force.*

If there is a resistance F' to the force F in the opposite direction, and if the acceleration due to F' is f', we have the resultant acceleration $f - f' = \frac{R}{mv} - \frac{F'}{m}$. Hence

$$v = \frac{R}{m(f - f') + F'}.$$

When $f = f'$ there is no resultant acceleration and v is uniform and *a maximum.*

Hence, *the greatest velocity which a force working at the rate R can produce against an opposing force F′ is equal to* $\frac{R}{F'}$.

EXAMPLES.

(1) *Find the work done against gravity in drawing a car of* 2.5 *tons* (2240 *lbs.*), *loaded with* 30 *passengers of* 154 *lbs. each, up an incline the ends of which differ in level by* 120 *feet, and also find the horse-power if the time is half an hour.*

Ans. 1226.400 ft.-lbs.; 1.24 horse-power.

(2) *Express a horse-power and a force de cheval in C. G. S. absolute units.*

Ans. 7.47×10^9 ergs per sec.; 7.36×10^9 ergs per sec.

(3) *Find the force de cheval in terms of the horse-power.*

Ans. 0.987 H. P.

(4) *Find the horse-power in terms of the force de cheval.*

Ans. 1.014 force de cheval.

(5) *Find the horse-power of a machine which raises* 10 *tons* (2240 *lbs.*) 20 *feet in* 2 *minutes.*

Ans. 6.8 H. P.

(6) *If an engine consumes* 2 *pounds of coal per horse-power per hour, how many foot-pounds of work will it perform when consuming* 112 *pounds of coal?*

Ans. 110 880 000 ft.-lbs.

(7) *If a pressure of* 1 *ton* (2240 *lbs.*) *is exerted through* 10 *yards, how many foot-pounds of work are done; and if the work is done in half a minute, what is the horse-power?*

Ans. 67200 ft.-lbs.; 4.07 horse-power.

(8) *A pumping-engine is partly worked by a weight of* 2 *tons, which at each stroke of the pump falls through* 4 *ft. The pump makes* 10 *strokes per minute. How many gallons of water are lifted per minute by the weight from a depth of* 200 *ft.? Take a gallon at* 8.355 *lbs.*

Ans. 107.24 gallons.

(9) *Calculate the horse-power of an engine from the following data: stroke* 24 *in., diameter of piston* 16 *in.,* 100 *revolutions per min., average effective pressure in the cylinder* 60 *lbs. per sq. in.*

Ans. 146 horse-power.

(10) *In the transmission of power by a belt, the wheel carrying the belt is* 14 *feet in diameter and makes* 30 *revolutions per minute, the tension of the rope being* 100 *lbs. Find the horse-power transmitted.*

Ans. 4 horse-power.

(11) *What diameter of cylinder will develop* 50 *horse-power with a four-foot stroke,* 40 *revolutions per minute, and a mean effective pressure of* 30 *lbs. per square inch above the atmosphere, the engine being non-condensing?*

Ans. 21 inches.

(12) *The cylinder of an engine is* 12 *inches diameter by* 20 *inches long. Average pressure* 60 *lbs. per square inch,* 40 *horse-power. Find the rate of revolution.*

Ans. 58.4 revolutions per minute.

(13) *If the acceleration of a falling body be taken as unit of acceleration, 1 ton as unit of mass, 1 horse-power as unit rate of work, and 1 min. as unit of time, find the desired unit of length.*

Ans. 14.7 feet.

(14) *A mass of 50 lbs. is drawn on a smooth horizontal plane, the work being at the rate of 1/10 horse-power. Find the acceleration when the speed is 1 mile per hour.* ($g = 32.2$.)

Ans. 24.15 ft. per sec. in the direction of motion.

(15) *An engine is employed in lifting a weight of 112 pounds. If the engine is working at 5 H. P. and the weight has a speed of 5 ft. per sec., find its acceleration. At what H. P. must the engine work to lift the weight with a uniform speed of 1 ft. per sec.?* ($g = 32.2$.)

Ans. 158.1 ft.-per-sec. per sec.; $\frac{1}{5}$H. P.

(16) *Find the greatest speed an engine of 100 H. P. can give a train of 70 tons (2240 lbs.) mass on an incline of 1 in 100, friction being equivalent to a force of 8 pounds per ton.* ($g = 32.2$.)

Ans. 17.62 miles per hour.

(17) *A train weighing 75 tons ascends an incline of 1 in 800 with a uniform speed of 40 miles an hour. Assuming friction to be equivalent to a force of 6 pounds per ton, find the rate at which the engine is working.*

Ans. 70.4 H. P.

(18) *Check this statement: Fifty-five pounds mean effective pressure at 600 ft. piston speed gives 1 H. P. for each square foot of piston area.*

Efficiency—Mechanical Advantage.—In a machine the "moving force" F acts at the "point of application" and a "useful" resistance F' is overcome, or work is performed at some other point, called the "*working point.*" If there is no friction, the rate of work of the moving force F must always equal that of the resistance. Owing to friction it must always be greater.

The ratio of the rate of work of the "useful" resistance to the rate of work of the moving force is called the **efficiency** of the machine.

It must always be a fraction less than unity, and approaches unity the more perfect the machine and the less the friction. If we denote it by ϵ, and let v be the velocity of the moving force F, and v' the velocity of the resistance F', we have the efficiency

$$\epsilon = \frac{F'v'}{Fv}, \quad \text{or} \quad F' = \epsilon \frac{v}{v'} F.$$

If there is no friction, $\epsilon = 1$ and $Fv = F'v'$. The ratio $\epsilon\frac{v}{v'}$ is called the **mechanical advantage** of the machine.

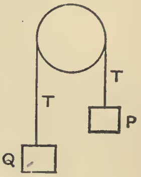
If F' is greater than F, v' must be less than v in nearly the same proportion, or, if friction is disregarded, in exactly the same proportion, that is, $\frac{F'}{F} = \frac{v}{v'}$.

Hence the familiar maxim that "*what is gained in force is lost in speed.*"

EXAMPLES.

(1) *Two masses of P lbs. and Q lbs. are hung by means of a perfectly flexible inextensible string over a smooth pulley. Disregarding friction and the mass of the pulley and rope, discuss the machine.* (See Ex. 13, page 9.)

Ans. If P is the larger mass, and if we disregard friction and the mass of the pulley, the moving force is $(P - Q)g$ poundals. The total mass moved is $(P + Q)$ lbs. If we denote the acceleration by f, we have



$$(P + Q)f = (P - Q)g, \quad \text{or} \quad f = \frac{(P - Q)g}{P + Q} \text{ ft.-per-sec. per sec.}$$

The space passed over from rest in t sec. is then, for P or Q,

$$s = \frac{1}{2}ft^2 = \frac{(P - Q)gt^2}{2(P + Q)} \text{ ft.}$$

The velocity starting from rest, at the end of t sec. is, for P or Q,

$$v = ft = \frac{(P - Q)gt}{P + Q} \text{ ft. per sec.}$$

The tension in the string on either side is

$$T = P(g - f) = Q(g + f) = \frac{2PQg}{P + Q} \text{ poundals or } \frac{2PQ}{P + Q} \text{ lbs.}$$

The pressure on the axle is the sum of the tensions (see page 9)

$$(P + Q)g - (P - Q)f, \quad \text{or} \quad \frac{4PQ}{P + Q} \text{ lbs.}$$

The work of P is the same as the work on Q, or

$$Ts = \frac{PQ(P - Q)gt^2}{(P + Q)^2} \text{ ft.-lbs.}$$

The power, or rate of work, is

$$\frac{PQ(P - Q)gt}{550(P + Q)^2} \text{ horse-power.}$$

The efficiency $\epsilon = 1$, since the rate of work of P equals the rate of work on Q.

The mechanical advantage $\epsilon \frac{v}{v'}$ is unity, since $\epsilon = 1$ and $\frac{v}{v'} = 1$.

(2) *If a thread of a screw makes 25 turns in 3 inches, and the arm is 24 inches, find the force to sustain a weight of 112 lbs. (friction disregarded); also the mechanical advantage.*

Ans. Let the moving force F make n revolutions per minute. Then its velocity is $v = \frac{2\pi \times 24 \times n}{60}$ inches per sec. The resistance F' moves 3/25 inch per revolution, or $v' = \frac{3n}{25 \times 60}$ inches per sec.

Since $Fv = F'v'$, we have

$$\frac{2\pi \times 24 \times n}{60}F = \frac{3n \times 112}{25 \times 60}, \quad \text{or} \quad F = \frac{112 \times 3}{2\pi \times 25 \times 24} = 0.089 \text{ lb.}$$

The efficiency $\epsilon = 1$. $F' = \frac{2\pi \times 25 \times 24}{3}F = 1256.6F$, or the mechanical advantage is 1256.6.

(3) *If the thread of a screw is inclined at an angle of 30° to the horizontal, the radius of the screw 9 inches, and the length of the arm 4 ft., find what force will sustain 1680 lbs.*

Ans. 100 lbs. Mechanical advantage 16.8.

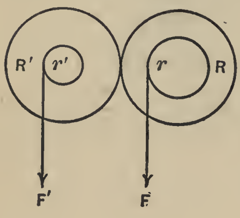
(4) *An endless screw whose pitch is 5/8 inch works in a worm-wheel having 16 teeth. The length of the handle is 10 inches. Find the mechanical advantage.*

Ans. Let $r =$ radius of wheel. Then $F \times 10 = F' \times r$, or $\frac{F'}{F} = \frac{10}{r}$. But $2\pi r = 16 \times \frac{5}{8}$, or $r = \frac{5}{\pi}$. Hence $\frac{F'}{F} = \frac{10\pi}{5}$, or about 44 to 7.

(5) *A wheel and axle is used to raise a bucket from a well. The radius of the wheel is 15 inches, and while it makes 7 revolutions the bucket, which weighs 30 lbs., rises 5½ ft. Find the smallest force to turn the wheel.*

Ans. 3 lbs. Mechanical advantage 10.

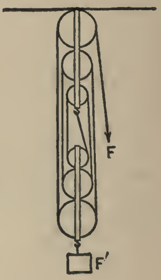
(6) *Two toothed wheels of radius R and R′. The force F is applied at a distance r from the centre of the first, and the resistance F′ is applied at a distance r′ from the centre of the second. Find the mechanical advantage. (Friction neglected.)*



Ans. We must have $Fv = F'v'$, or $\frac{F'}{F} = \frac{v}{v'}$. Let ω be the angular velocity of one wheel and ω' that of the other. Since the linear velocity of the point of contact is the same for both, we have $R\omega = R'\omega'$, or $\frac{\omega}{\omega'} = \frac{R'}{R}$. The velocity of F is $v = r\omega$, and of F', $v' = r'\omega'$. Hence

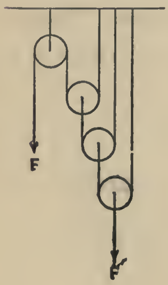
$$\frac{v}{v'} = \frac{r\omega}{r'\omega'} = \frac{rR'}{r'R} = \frac{F'}{F}. \quad \text{Therefore} \quad F' = \frac{rR'}{r'R}F.$$

(7) *In the system of pulleys shown in the figure find the mechanical advantage, neglecting friction.*



Ans. Whatever distance F' may be raised in a given time, each of the n cords will be shortened by that much. We have then $nv' = v$, or $\frac{v}{v'} = \frac{F'}{F} = n$, where n is the number of cords at the lower block.

(8) *In the system of pulleys shown in the figure find the mechanical advantage, neglecting friction.*

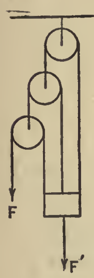


Ans. Whatever distance F passes through in a given time,

the first movable pulley passes through $\frac{1}{2}$ that distance;
" second " " " " $\frac{1}{4}$ " "
" third " " " " $\frac{1}{8}$ " "
" nth " " " " $\frac{1}{2^n}$ " "

Hence $v' = \frac{1}{2^n}v$, or $\frac{v}{v'} = \frac{F'}{F} = 2^n$, where n is the number of movable pulleys.

(9) *In the system of pulleys shown in the figure find the mechanical advantage, neglecting friction.*



Ans. Whatever distance F' rises in a given time, each cord will be shortened by that distance. Let this distance be d.

The 1st movable pulley will descend through distance d;

" 2d through dist. $(2 + 1)d$;

" 3d " " $2(2 + 1)d + d = (2^2 + 2 + 1)d$;

" 4th " " $2[2(2 + 1)d + d] + d = (2^3 + 2^2 + 2 + 1)d$;

" nth " " $(2^{n-1} + 2^{n-2} + \dots 2 + 1)d$.

The force F will descend through twice the distance of the nth movable pulley $+ d$, or

$$2(2^{n-1} + 2^{n-2} + \dots 2 + 1)d + d = d(2^n + 2^{n-1} + \dots 2^2 + 2 + 1)$$
$$= d(2^{n+1} - 1).$$

Hence $\frac{v}{v'} = \frac{2^{n+1} - 1}{1} = \frac{F'}{F}$, where n is the number of movable pulleys.

(10) *A man weighing* 175 *lbs. is lowered into a well by means of a windless the arm and axle of which are* 30 *inches and* 8 *inches diameter. Find the force which must be applied to let him down with uniform velocity.*

Ans. $46\frac{2}{3}$ lbs.

(11) *Suppose we have four pulleys as in Ex.* (8), *three movable and one fixed, and that the weight is a man weighing* 160 *lbs. Find what pull the man must exert in order to raise himself.*

Ans. 20 lbs.

(12) *A weight of* 336 *lbs. is raised* 3 *feet by means of a single movable pulley the block of which has three sheaves. Find the force and the distance through which it acts.*

Ans. 56 lbs.; 18 ft.

(13) *In the system of pulleys in Ex.* (7), *if the block weighs* 8 *lbs. and there are three pulleys in the lower block, find the weight which a force of* 20 *lbs. can support.*

Ans. 112 lbs.

(14) *Find the mechanical advantage in a system of three pulleys similar to Ex.* (9).

Ans. $\frac{F'}{F} = 7$.

(15) *The thread of a screw makes* 12 *turns in a foot of length. The moving force is applied at the end of an arm* 2 *feet long. It is found that when this force is* 30 *lbs. it can just raise a weight of* 1200 *lbs. What portion of the moving force is expended against friction, and how many foot-pounds of work are performed by the moving force when the weight is raised* 2 *feet?*

Ans. 22 lbs.; 2400 ft.-lbs. of work.

CHAPTER V.

ENERGY. KINETIC ENERGY.

ENERGY. KINETIC ENERGY. ILLUSTRATIONS OF KINETIC ENERGY. BODY MOVING IN A RESISTING MEDIUM.

Energy.—Work, as we have seen (page 41), is done by a uniform force upon a particle when the particle has a component displacement in the direction of the force, and work is done by a particle against a uniform force when the particle has a component displacement in a direction opposite to that of the force. When a particle is able to thus do work against a force it is said to **possess energy**, and the work it is capable of doing is called its **energy.**

The unit of energy is therefore the unit of work (page 42).

Kinetic Energy.—The work which a particle is able to do by reason of its velocity is called its **kinetic energy.**

Determination of Kinetic Energy.—Let a particle of mass m be at rest, and let it be acted upon by a uniform force F in any direction, and at the end of any time acquire the velocity v in that direction. The uniform force F causes a uniform acceleration

$$f = \frac{F}{m}.$$

The path is a straight line, and the distance described in the path, which in this case is the displacement, is (page 51, Vol. I, *Kinematics*)

$$s = \frac{v^2}{2f}.$$

This displacement is in the direction of the force. Inserting the value of f, we have

$$s = \frac{mv^2}{2F}, \quad \text{or} \quad Fs = \frac{1}{2}mv^2,$$

where by definition Fs is the work of the force F in giving the particle the velocity v, starting from rest. This work is evidently also the work which a particle of mass m moving with velocity v can do while coming to rest against a uniform opposing force F. It is therefore the *kinetic energy* of the particle, or the work the particle is capable of doing by virtue of its velocity.

Hence, *the kinetic energy of a particle is equal to one half the product of its mass by the square of its velocity.*

If then v_1 is the initial and v the final velocity of a particle moving in a straight line, then $\frac{1}{2}mv_1^2$ is the work the particle can do by virtue of its initial velocity, and $\frac{1}{2}mv^2$ is the work it can do by virtue of its final velocity. If v_1 is greater than v, then the difference $\frac{1}{2}mv_1^2 - \frac{1}{2}mv^2$ is the work which the particle has done against uniform opposing force. This work is negative (—). If v is greater than v_1, then $\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2$ is the work done by the uniform force upon the particle. This work is positive (+). We have then in general for uniform force and path a straight line

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = W. \quad \ldots\ldots\ldots \quad (1)$$

That is, *the gain or loss of kinetic energy is equal to the work done by or against the uniform force.*

This work we see is independent of the time and depends simply upon the velocities v_1 and v.

The same holds true whatever the path, whatever the time or number of forces and whether the forces are uniform or variable.

Thus let any number of variable forces act upon a particle moving in any path. The forces acting upon the particle at any point of its path can be resolved into a resultant normal and tangential component. The normal component does no work.

Since the work of the resultant is equal to the algebraic sum of the works of the components (page 42), the work of the variable forces is equal to the work of the resultant variable tangential component.

If we divide the path into an indefinitely large number of indefinitely small displacements, this tangential component may be considered as uniform during each displacement. Let then F_1, F_2, F_3, etc., be the uniform tangential components during the small displacements s_1, s_2, s_3, etc., and let v_1 be the initial velocity and v_2, v_3, etc., be the velocities after the successive displacements s_1, s_2, etc. Then if v is the final velocity, we have

$$\begin{aligned}
F_1s_1 &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2;\\
F_2s_2 &= \frac{1}{2}mv_3^2 - \frac{1}{2}mv_2^2;\\
F_3s_3 &= \frac{1}{2}mv_4^2 - \frac{1}{2}mv_3^2;\\
&\ldots\ldots\ldots\ldots;\\
F_ns_n &= \frac{1}{2}mv^2 - \frac{1}{2}mv^2_{n-1}.
\end{aligned}$$

If W is the total work, we have by summation

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2. \quad \ldots\ldots\ldots \quad (1)$$

We see then that equation (1) holds in all cases, whatever the time or path or number of forces, and whether the acting forces are uniform or variable. (The derivation of equation (1) by calculus is given on page 46.) If $v_1 = 0$, we have $W = \frac{1}{2}mv^2$.

In general, then, *kinetic energy gives the work a moving particle can do against force by virtue of its velocity, or the work done by force in giving it that velocity. It is always given by* $\frac{1}{2}mv^2$ *whatever the time, path or forces.*

The gain or loss of kinetic energy is equal to the work done by or against the acting forces whatever the time, path or forces.

The product $\frac{1}{2}mv^2$ gives the work in foot-poundals if m is taken in pounds and v in feet per second. If we wish the work in gravitation measure, or in foot-pounds, we must divide by g in feet-per-sec. per sec.

If we take m in grams and v in centimetres per second, the product $\frac{1}{2}mv^2$ gives the work in *ergs*, or dyne-centimetres. If we wish the work in gravitation measure, or in gram-centimetres, we must divide by g in centimetres-per-sec. per sec.

Cor. If the path is a circle, we have $v = r\omega$, where ω is the angular velocity and r is the radius. The kinetic energy is then given by $\frac{1}{2}mr^2\omega^2$, and we have

$$W = \frac{1}{2}mr^2\omega^2 - \frac{1}{2}mr^2\omega_1^2.$$

Illustrations of Kinetic Energy.—We have seen (page 33) that if a particle of mass m has at any instant the velocity v_1 in any direction, the uniform force F *always opposed to that direction* which would bring it to rest in the time t is equal to the rate of change of momentum, or $F = \frac{mv_1}{t}$. But the space described in coming to rest is $\frac{v}{2}t = s$. Hence $-Fs = \frac{1}{2}mv_1^2$. Here we see that the work against the force necessary to bring the particle to rest is the kinetic energy, no matter what the time or path may be.

Again, let a particle of mass m start from rest with a constant rate of change of speed a, and describe the distance s. Then the final speed attained, whatever the path, is given (page 28, Vol. I, *Kinematics*) by $v^2 = 2as$. If we multiply by $\frac{m}{2}$, we have $\frac{1}{2}mv^2 = mas$. But ma is the tangential force F_t, and F_ts is the work (page 44). Here we see that the kinetic energy $\frac{1}{2}mv^2$ gives the work done by the force in imparting the velocity v.

Again, let a particle of mass m be acted upon by a force or acceleration proportional to the distance of the particle from a fixed point, and let the particle move from a distance R to a distance r. Then we have (page 104, Vol. I, *Kinematics*), if the particle starts from rest, whatever the path may be,

$$v^2 = \frac{a'}{r'}(R^2 - r^2),$$

where a' is the rate of change of speed at a given distance r' along the path, and R and r are measured along the path.

If we multiply by $\frac{m}{2}$, we have $\frac{1}{2}mv^2 = \frac{ma'}{2r'}(R^2 - r^2) = \frac{F'}{2r'}(R^2 - r^2)$, where F' is the force at a distance r', and we have shown (page 47) that this is the work done by the force in giving the velocity v.

Again, let a particle of mass m be acted upon by a force or acceleration inversely proportional to the square of the distance from a fixed point, and let the particle move from a distance R to a distance r. Then we have (page 99, Vol. I, *Kinematics*), if the particle starts from rest, whatever the path may be,

$$v^2 = 2a'r'^2\left(\frac{1}{r} - \frac{1}{R}\right),$$

where a' is the rate of change of speed at a given distance r' along the path, and R and r are measured along the path.

If we multiply by $\frac{m}{2}$, we have

$$\frac{1}{2}mv^2 = ma'v'^2\left(\frac{1}{r} - \frac{1}{R}\right) = F'r'^2\left(\frac{1}{r} - \frac{1}{R}\right),$$

where F' is the force at a distance r', and we have shown (page 47) that this is the work done by the force in giving the velocity v.

Thus we see that in all cases $\frac{1}{2}mv^2$, or the kinetic energy, gives the work a particle can do in coming to rest, or the work necessary to give it the velocity v no matter what the law of force, time or path.

We have then, generally,

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = W,$$

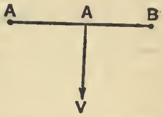
where v_1 and v are the initial and final velocities.

Hence, *the gain or loss of kinetic energy gives the work done by or against the acting forces;* and

Work done by a force is positive (+), *work against a force is negative* (−).

[**Body Moving in a Resisting Medium—Coefficient of Resistance.**—Let Δ be the density or mass of a unit of volume of the medium, and δ the density of the body, considered homogeneous.

Consider first the case of a plane surface AB moving in a direction at right angles to the surface. Let v be the velocity at any instant, and A the area of the surface.



In any indefinitely small time dt every particle of the medium which comes in contact with the plane has its velocity in the direction of motion increased from zero to v.

The corresponding pressure normal to the surface is then the rate of change of momentum (page 33), or the normal resistance to motion is $\frac{mv}{dt}$, where m is the mass of the displaced medium in the short time dt.

If the resistance is considered constant for the very short time dt, the distance traversed by the plane is $\frac{vdt}{2}$, and the mass of medium displaced is $m = \frac{\Delta Avdt}{2}$, where A is the area of surface at right angles to the motion.

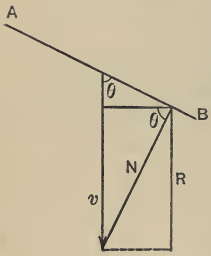
Therefore the resistance is

$$\frac{mv}{dt} = \frac{\Delta Av^2}{2}.$$

The same result is thus obtained by the principle of energy: To impart the velocity v to the mass m of the medium requires a work of $\frac{mv^2}{2}$. The distance passed through in a short time dt is $\frac{vdt}{2}$. The constant pressure for that time is the work divided by the distance, or $\frac{mv}{dt}$. The volume of medium moved is the area A at right angles to the motion multiplied by the distance, or $\frac{Avdt}{2}$, and its mass is $m = \frac{\Delta Avdt}{2}$. Hence the resistance is, as before,

$$\frac{mv}{dt} = \frac{\Delta Av^2}{2}.$$

Consider now a plane moving in a direction oblique to the surface.



Let AB move vertically with a velocity v and an inclination θ to the vertical.

The velocity normal to the surface is $v \sin \theta$, and hence the *normal pressure* is as before

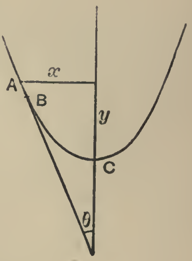
$$\frac{\Delta Av^2 \sin^2 \theta}{2} = N.$$

The component of this pressure in the direction of motion is

$$R = N \sin \theta = \frac{\Delta Av^2 \sin^3 \theta}{2} = \frac{\Delta}{2} A \sin \theta \times v^2 \sin^2 \theta.$$

But $A \sin \theta$ is the projection of the surface at right angles to the motion.

Consider next a solid of revolution moving in the direction of its axis CD. Let AB be any element ds of the generating curve, making the angle θ with CD, and let its co-ordinates be x and y.



Then the projection of AB at right angles to the motion is $2\pi x dx = A \sin \theta$; and since $\sin \theta = \frac{dx}{ds}$, we have for the resistance in the direction of motion

$$\frac{\Delta}{2} \times 2\pi x dx \times v^2 \frac{dx^2}{ds^2} = \pi \Delta v^2 x dx \frac{dx^2}{ds^2}.$$

The mass of the solid is $\pi\delta \int x^2 dy$.

The retardation of the body or minus acceleration is then

$$-f = \frac{\text{force}}{\text{mass}} = \frac{\pi \Delta v^2 \int x dx \frac{dx^2}{ds^2}}{\pi\delta \int x^2 dy}.$$

We see then that

$$f = -cv^2,$$

or the retardation is directly proportional to the square of the velocity. The constant c is called the *coefficient of resistance*, and is given by

$$c = \frac{\varDelta}{\delta} \frac{\int x dx \frac{dx^2}{ds^2}}{\int x^2 dy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the effect of friction has been disregarded, it is customary to put

$$f = -\zeta cv^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ζ is an experimental constant and c is given by (1).

For a sphere we have

$$x^2 + y^2 = r^2, \quad \frac{dx}{dy} = -\frac{y}{x}, \quad ds = \sqrt{dx^2 + dy^2},$$

and hence

$$\frac{dx^2}{ds^2} = \frac{1}{1 + \frac{dy^2}{dx^2}} = \frac{1}{1 + \frac{x^2}{y^2}} = \frac{y^2}{r^2}.$$

Substituting in (1), we have

$$c = \frac{\varDelta}{\delta r^2} \frac{\int_0^{+r} x(r^2 - x^2)dx}{\int_{-r}^{+r} (r^2 - y^2)dy} = \frac{3\varDelta}{16\delta r}.$$

For a cone we have, if r is the radius of base and h the height,

$$y : x :: h : r, \quad \text{or} \quad ry = hx, \quad \frac{dx}{dy} = \frac{r}{h}, \quad ds = \sqrt{dx^2 + dy^2},$$

and hence

$$\frac{dx^2}{ds^2} = \frac{1}{1 + \frac{dy^2}{dx^2}} = \frac{1}{1 + \frac{h^2}{r^2}}.$$

Substituting in (1),

$$c = \frac{\varDelta}{\delta} \frac{\int_0^r \frac{xdx}{1 + \frac{h^2}{r^2}}}{\int_0^r \frac{h}{r} x^2 dx} = \frac{3\varDelta r^2}{2\delta h(r^2 + h^2)}.$$

If the cone terminates in a cylinder of length l, we have

$$c = \frac{3\varDelta r^2}{2\delta(3l + h)(r^2 + h^2)}.$$

These are the values of c used in pages 110–113, Vol. I, *Kinematics*.

For iron in water we may take $\frac{\delta}{\varDelta} = 7.2$;

" " " air " " " $\frac{\delta}{\varDelta} = 5983.28$;

" mist or rain in air " " " $\frac{\delta}{\varDelta} = 813.82$;

" lead in water " " " $\frac{\delta}{\varDelta} = 11.35$;

" " " air " " " $\frac{\delta}{\varDelta} = 9423.61$.

These are the values of $\frac{\delta}{\varDelta}$ assumed in page 113, Vol. I, *Kinematics.*

EXAMPLES.

(1) *A fly-wheel has a mass of* 30 *tons, which is to be considered as distributed around the circumference of a circle* 8 *ft. in radius; it makes* 20 *revolutions per minute. Find its kinetic energy.*

Ans. 292900 ft.-lbs.

(2) *A ball weighing five ounces and moving with a velocity of* 1000 *ft. per sec. strikes an obstacle, and after piercing it moves on with a velocity of* 400 *feet per sec. Find the energy lost.*

Ans. 131250 ft.-poundals or $\frac{131250}{g}$ ft.-lbs.

(3) *What constant force will bring a car of* 5 *tons, moving with a speed of* 6 *miles per hour, to rest in* 20 *feet?*

Ans. 21683 poundals or $\frac{21683}{g}$ pounds.

(4) *How far will a car run on level rails if it has an initial speed of* 10 *miles an hour and friction is* 1/20 *of the weight?*

Ans. 66.8 feet.

(5) *If an ounce bullet leaves a gun with a velocity of* 800 *ft. per sec., the barrel being* 3 *ft. long, what is the accelerating force, supposing it uniform?*

Ans. 6666 poundals or $\frac{6666}{g}$ pounds.

(6) *A shot of* 1000 *lbs. moving at* 1600 *ft. per sec. strikes a fixed target; how far will the shot penetrate if the average pressure is* 12000 *tons?*

Ans. 1.49 feet.

(7) *A train of* 200 *tons, starting from rest, acquires a speed of* 40 *miles an hour in three minutes. What is the effective moving force, assuming it uniform?*

Ans. 2.03 tons.

(8) *A bullet weighing* 2½ *oz. leaves a gun with a velocity of* 1550 *ft. per sec.; the length of barrel is* 2½ *feet. Find the average accelerating force.*

Ans. 2332 lbs.

(9) *A ball-player catches a ball moving at* 50 *ft. per sec. The mass of the ball is* 4 *oz. If the space during which the ball is brought to rest is* 6 *inches, what is the average pressure on the hands? What is the time of stoppage?*

Ans. 19.4 lbs.; 1/50 sec.

(10) *A fly-wheel has a mass of* 30 *tons, which may be considered as distributed along the circumference of a circle* 8 *ft. in radius. It starts from rest and, under the action of a constant force applied at the extremity of a crank* 18 *inches long, acquires a speed of* 20 *revolutions per minute in one minute. Find the force on the crank.*

Ans. 3100 lbs.

(11) *A heavy body is projected up an incline rising* 1 *in* 100 ; *the friction against the plane is one tenth of the pressure. Find the distance it will travel before being reduced to rest, the velocity of projection being* 121 *ft. per sec.*

Ans. 2067 feet.

(12) *Find the tension on a rope which draws a carriage of* 8 *tons up a smooth incline of* 1 *in* 5, *and causes an increase of velocity of* 3 *ft.-per-sec. per sec.*

Ans. 169165 poundals or $\frac{169165}{g}$ lbs.

(13) *If on the same incline the rope breaks when the carriage has a velocity of* 48.3 *ft. per sec., how far will the carriage continue to move up the incline?*

Ans. 181 feet.

(14) *A mass P, after falling freely through h ft., begins to pull up a heavier mass Q by means of a string passing over a smooth pulley. Find the height to which Q will be lifted.*

Ans. $\frac{ph}{q-p}$ feet.

(15) *The tractive force of an engine is P tons. If the weight of engine and train is W tons and the frictional resistance n lbs. per ton, show that in going up a p-per-cent grade the velocity acquired in t seconds from rest will be Qgt ft. per sec., and the energy* $0.5\,WQ^2gt^2$ *ft.-tons, where* $Q = \frac{P}{W} - \frac{p}{100} - \frac{n}{2240}$.

(16) *In a brake test, a train moving* 22 *miles an hour on a down grade of* 1 *per cent was stopped in* 91 *ft. There was* 94 *per cent of the train braked. Taking the frictional resistance as* 8 *lbs. per ton of* 2000 *lbs., find the net brake resistance per ton, and the grade to which this is equivalent.* ($g = 32$.)

Ans. 393 lbs., equivalent to 19.65 per cent grade.

(17) *An engine exerts on a car weighing* 20000 *lbs. a net pull of* 2 *lbs. per ton of* 2000 *lbs. Find the energy stored in the car after going* 2½ *miles.*

Ans. 264,000 ft.-lbs.

(18) *If shunted onto a level side track where the frictional resistance is* 10 *lbs. per ton, how far will it run?*

Ans. One-half mile.

(19) *If the side track has a one-per-cent grade?*

Ans. One-sixth mile.

(20) *What effect has the recoil of a gun upon its range?*

Ans. Let $M =$ the mass of the gun and V its velocity of recoil;
$m =$ " " " " ball " v " " " ".

If these velocities are produced in a short time t, the mean pressure on the gun is $\frac{MV}{t}$, and on the ball $\frac{mv}{t}$. These must be equal, and hence

$$MV = mv, \quad \text{or} \quad V = \frac{mv}{M}.$$

The work on the gun is then $\frac{1}{2}MV^2 = \frac{1}{2}\frac{m^2v^2}{M}$. The work on the ball is $\frac{1}{2}mv^2$. Let $w =$ the entire work of the powder. Then

$$\frac{1}{2}mv^2 + \frac{1}{2}\frac{m^2v^2}{M} = w, \quad \text{or} \quad v = \sqrt{\frac{2w}{m + \frac{m^2}{M}}}.$$

If w and m are constant, the velocity v of the ball will increase as M increases. But as M increases, the velocity of recoil V decreases. Thus v increases as the recoil diminishes. If M is infinite, we have no recoil at all and $v = \sqrt{\frac{2w}{m}}$, or all the work is done on the ball. If $M = nm$, we have

$$v = \sqrt{\frac{2w}{m}\frac{n}{n+1}}.$$

Since $\frac{n}{n+1}$ is always less than unity, we see that v diminishes as n diminishes. Theoretically, then, a gun shoots farther the greater its mass and the less its recoil.

CHAPTER VI.

KINETIC FRICTION.

FRICTION. ADHESION. KINDS OF FRICTION. REACTION OF A ROUGH CURVE OR SURFACE. TRANSLATION OF A BODY ON ANY CURVE OR SURFACE. COEFFICIENT OF KINETIC SLIDING FRICTION. ANGLE OF KINETIC FRICTION. LAWS OF KINETIC SLIDING FRICTION. MOMENT AND WORK OF FRICTION. KINETIC FRICTION OF PIVOTS, AXLES, ROPES, ETC. EXPERIMENTAL DETERMINATION OF COEFFICIENTS OF KINETIC SLIDING FRICTION. FRICTION-BRAKE TEST. WORK OF AXLE-FRICTION. TABLE OF COEFFICIENTS OF KINETIC FRICTION.

Friction.—Every natural surface offers a resistance to the motion of a body upon it. Part of this resistance is due to **adhesion** between the body and surface, and part is due to **friction.**

Friction, then, is always a retarding force or resistance, and *acts always in a direction opposite to that in which the body moves.*

When one surface moves upon another, the surfaces in contact are compressed and projecting points and irregularities are bent over, broken off, rubbed down, etc.

The resistance due to friction, therefore, evidently depends upon the materials of which the surfaces are composed, and also upon the roughness or smoothness of the surfaces in contact.

It may also evidently vary for the same surfaces, according to their condition or state or material constitution.

Thus it may not be the same for surfaces of dry wood or iron as for the same surfaces under the same condition when wet. It may not be the same for two surfaces of wood with their fibres parallel as for the same surfaces under the same conditions when their surfaces are not parallel.

Unguents also have a great influence. Such fluid or semi-fluid unguents as oil, tallow, etc., fill up interstices and diminish the effect of irregularities of surfaces; or a film of unguent may be interposed between the surfaces and thus the resistance of friction greatly diminished.

Adhesion.—We must not confound the resistance due to friction with that due to adhesion. Adhesion is that resistance to motion which takes place when two different surfaces come in contact at many points without pressure. Adhesion increases with the area of the surface of contact and is independent of the pressure, while, as we shall see (page 67), friction increases with the pressure and is in general independent of the area of surface of contact.

If, however, the pressure is great, adhesion may be neglected compared to friction, and the resistance to motion is then practically that due to the friction only.

When the surfaces in contact are of the same kind, we call the resistance to motion **cohesion**; when of different kinds, **adhesion.**

Kinds of Friction.—Surfaces may slide or roll on one another. We distinguish accordingly **sliding friction** and **rolling friction.**

It is also found by experiment that the friction which just prevents motion is greater than that which exists after actual motion takes place. The friction which just prevents motion is called friction of repose or quiescence, or **static friction.** The friction which exists after actual motion takes place is called friction of motion, or **kinetic friction.**

We have then two kinds of static friction, viz., **static sliding friction** and **static rolling friction.**

We have also two kinds of kinetic friction, viz., **kinetic sliding friction** and **kinetic rolling friction.**

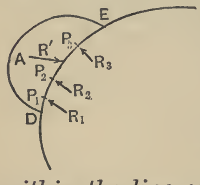
We have to do in this portion of our work with kinetic friction only. We have already treated static friction in Chap. IX, Vol. II, *Statics.*

Reaction of a Curve or Surface.—When a particle is in contact with a rigid material curve or surface, the pressure which the curve or surface exerts upon the particle is called the **reaction** of the curve or surface.

If then we introduce this reaction as an additional force in combination with all the other forces acting upon the particle, we can remove the curve or surface and consider the motion of the particle under the action of all the other forces and of this reaction.

The reaction of the curve or surface is a force internal to the system, or a *stress* (page 37). All the other forces acting upon the particle we may then call *external forces.*

Translation of a Body on any Curve or Surface.—Let a rigid body ADE move *by sliding* on any curve or surface, and touch it at many points P_1, P_2, P_3, etc. Let the reactions at these points be R_1, R_2, R_3, etc., and let the resultant of all the external forces be R' acting at A.



Let the line of direction R' intersect the curve or surface at P. Then, if the curve or surface resists by pressure only, this point P, for sliding motion only or translation, *must evidently fall within the line or surface of contact DE.* For if it falls outside, then, since the resultant reaction must be inside, the body will rotate, and we have sliding and rolling, and not translation only.

Since we can replace the curve or surface at any point of the base DE by its reaction at that point, *we can treat the entire body for sliding only, as a particle of equal mass placed at any one of its points of contact and acted upon by the reaction at that point and all the other forces and reactions, considered as external forces. The motion of this particle is the same as that of the body.*

Since we are dealing now with translation only, we can then consider all cases as *the motion of a particle on a curve or surface.*

Coefficient of Kinetic Sliding Friction.—When one surface slides upon another, the sum of the frictions at every point of contact is the *total friction*, and the sum of the normal pressures at every point of contact is the *total normal pressure.*

The ratio of the total friction to the total normal pressure when motion, either sliding or rolling, is *just about to begin*, we have called (page 189, Vol. II, *Statics*) the **coefficient of static friction,** either of sliding or rolling.

The same ratio, *after motion has taken place*, is called the coefficient of kinetic friction, either of sliding or rolling.

We denote the coefficient of friction in general by μ. We have then in all cases

$$\mu = \frac{F}{N}, \quad \text{or} \quad F = \mu N,$$

where F is the total friction and N the total normal pressure when motion, either sliding or rolling, is just about to begin, or else has taken place.

We have considered static friction in Chap. IX, Vol. II, *Statics.* We have to do in this portion of the work with kinetic friction only.

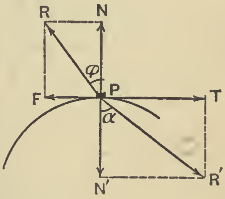
When a body slides upon a rough curve or surface, the motion of the body is that of any one of its points, and, as we have just seen, we can replace the body by a particle of equal mass at any point of contact, and acted upon by the reaction at that point, and all the other forces and reactions considered as external forces. We have then in this case also

$$\mu = \frac{F}{N}, \quad \text{or} \quad F = \mu N,$$

where F is the friction and N the normal pressure at the point of contact considered.

Angle of Kinetic Friction.—Suppose a body sliding on a rough curve or surface. We can then replace it by a particle of equal mass at any point of contact P. Let the reaction of the curve or surface be R, making the angle ϕ with the normal.

Let the resultant of all other forces acting upon the body be R', making the angle α with the normal. We can resolve R' into a normal component $N' = R' \cos \alpha$ in the plane of R and R', and a tangential component $T = R' \sin \alpha$ in the same plane. We can also resolve the reaction R into the normal component $N = R \cos \phi$, and the tangential component $F = R \sin \phi$ in the same plane.



If the body moves on the curve or surface, N must be always equal and opposite to N'. It is evident, then, that PT is the direction of motion, and F is the resistance acting opposite to the direction of motion, or the *friction* at the point of contact P; while the tangential resultant $T - F$ is the moving force. We have, therefore,

$$F = R \sin \phi, \quad N = R \cos \phi,$$

and

$$\tan \phi = \frac{F}{N}.$$

We call the angle ϕ which the reaction R makes with the normal in the plane of R and R' the **angle of kinetic friction.**

Hence, *the tangent of the angle of kinetic friction is equal to the ratio of the friction to the normal component of the reaction.*

A normal to a surface at any point must lie in the radius of curvature.

A normal to a curve at any point may have any direction in a plane through that point at right angles to the tangent at that point. The "normal component of the reaction" must then always be understood to mean the normal component *in the plane of the reaction R and the resultant R' of all other forces acting upon the body.*

We have just seen that the coefficient of kinetic friction is given by

$$\mu = \frac{F}{N}.$$

We have then

$$\mu = \frac{F}{N} = \tan\phi, \quad \text{or} \quad F = \mu N = N\tan\phi.$$

That is, *the coefficient of kinetic friction is equal to the tangent of the angle of kinetic friction.*

It is also evident that if $T - F = 0$ there is equilibrium. If there is no equilibrium $T - F$ must be greater or less than zero, and since N' is always equal to N, *the angle α must always be greater or less than ϕ*, or $\frac{T}{N'} \lessgtr \mu$.

Reaction of a Smooth Curve or Surface.—If the curve or surface is smooth there is no friction, and $F = 0$, $\phi = 0$.

A smooth curve or surface, then, *is one whose reaction is normal.* It is incapable of offering resistance to motion in any other than a normal direction. In such case we have then $R = N$, or the reaction is equal to the normal reaction.

Laws of Kinetic Sliding Friction.—The laws of kinetic sliding friction are the same as for static, as given on page 191, Vol. II, *Statics.* Within the practical limits there indicated we assume that

$$\mu = \frac{F}{N}$$

is constant for the same two surfaces in the same condition, whatever the area of contact and whatever the total normal pressure. To this we may add, whatever the velocity, within certain limits. If in any case these limits are exceeded, recourse must be had to special experiments for the value of μ for that case.

Moment and Work of Friction.—Since, then, $\mu = \frac{F}{N}$ is constant for the same two surfaces in the same condition, the friction F *at any point of contact* is given by $F = \mu N$, *where N is the normal pressure at that point.* This friction is always opposite in direction to the motion and tangent to the surface at the point.

The moment of the total friction with reference to any point is then equal to the algebraic sum of the moments of the frictions at every point of contact. If all these frictions have the same lever-arm, we may then consider the total friction as acting at any point of contact. Thus for an axle in a bearing we may take the total friction as acting at any point of contact tangent to the axle and opposite to the direction of motion.

The work done against friction is also evidently equal to the sum of the works done against the frictions at every point of contact. If the distances passed through by every point of contact of one surface, relatively to the other, are the same, we may again consider the total friction as acting at any point of contact. Thus for an axle in a bearing we may take the total friction as acting at any point of contact tangent to the axle and opposite to the direction of motion. The work done against friction is then the product of the total friction by the distance passed through with reference to the bearing of any point of the axle.

Kinetic Friction of Pivots, Axles, Ropes, etc.—The application of the equation

$$\mu = \frac{F}{N} = \tan\phi, \quad \text{or} \quad F = \mu N = N\tan\phi,$$

to pivots, axles, ropes, etc., is then precisely the same as for static friction (Chap. IX, Vol. II, *Statics*). We have only to let μ stand for the coefficient of kinetic instead of static sliding friction.

With this change we have in each case the same value for the friction and moment of the friction as already given in the chapter cited.

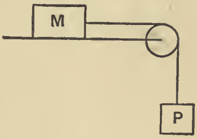
Rigidity of Ropes.—The influence of the rigidity of ropes has also been discussed in Chap. IX, Vol. II, *Statics*. The same results hold good in this portion of the work.

Experimental Determination of Coefficients of Kinetic Sliding Friction.—We may determine the coefficient of sliding friction by means of various contrivances, some of which we shall now describe.

1. **By Moving Sled and Weight.**—Let a sled rest upon a horizontal plane and be dragged along by means of a string passing over a fixed pulley to the end of which a weight is attached. In order to obtain coefficients for different substances, the runners and the plane can be covered with the materials to be experimented upon.

In such an apparatus the mass of string and pulley and friction of string and pulley on its axle, as well as the rigidity of the string, should all be insignificant, or else they must be taken into account.

If we disregard, then, mass of string and pulley, friction of string and pulley, and rigidity of string, and let M be the mass of the sled and P the mass of the weight in pounds, the normal pressure of M on the plane is Mg, and the weight is Pg poundals.



When motion *just begins* we have then *for the coefficient of static sliding friction*, since the friction F is equal to the weight Pg,

$$\mu = \frac{F}{N} = \frac{Pg}{Mg} = \frac{P}{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If Pg is greater than the weight just necessary to start the sled, then the moving force (see Ex. 12, page 8) is $Pg - \mu Pg$. where μ is the coefficient of kinetic sliding friction. The mass moved is $P + M$. Hence the uniform acceleration is, by the equation of force (page 2),

$$f = \frac{Pg - \mu Mg}{B + M}, \quad \text{or} \quad \mu = \frac{P}{M} - \frac{(P + M)f}{Mg}.$$

But for uniformly accelerated motion we have the distance described starting from rest, in any time t (page 51, Vol. I, *Kinematics*),

$$s = \tfrac{1}{2}ft^2, \quad \text{or} \quad f = \frac{2s}{t^2}.$$

Substituting this, we have for the coefficient of kinetic sliding friction

$$\mu = \frac{P}{M} - \frac{2(P+M)s}{Mgt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We see then from (1) and (2) that the coefficient of kinetic sliding friction *is less than the coefficient of static sliding friction.*

From equation (2), by noting the time t and the space s described in that time, we can calculate the value of μ.

2. **By Sled on Inclined Plane.**—If we place the sled on an inclined plane, and then gradually incline the plane, and note the angle ϕ at which the sled *just begins* to slide, this angle is the angle of repose, and, as we have seen (page 190, Vol. II, *Statics*), the tangent of this angle *is the coefficient of static sliding friction.*

If h is the altitude and b the base of the plane, we have then for the coefficient of static sliding friction

$$\mu = \tan\phi = \frac{h}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we allow the sled to slide down a plane whose angle α is greater than this, the moving force is $Mg \sin\alpha - \mu Mg \cos\alpha$, where μ is the coefficient of kinetic sliding friction. The mass moved is M. Hence

$$f = \frac{2s}{t^2} = g(\sin\alpha - \mu\cos\alpha),$$

or

$$\mu = \tan\alpha - \frac{2s}{gt^2\cos\alpha}.$$

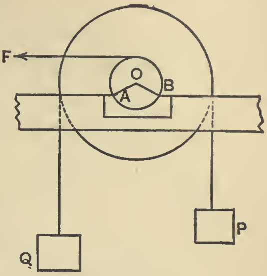
If h is the altitude, l the length, and b the base of the plane, $\tan\alpha = \frac{h}{b}$, $\cos\alpha = \frac{b}{l}$, and

$$\mu = \frac{h}{b} - \frac{2sl}{gbt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, we see from (1) and (2) that the coefficient of kinetic *is less than the coefficient of static sliding friction.*

From equation (2), by noting the distance s described in the time t, we can calculate μ. This apparatus is free from the sources of error of the first, due to mass of rope and pulley, friction of rope and pulley, and rigidity of rope.

3. By Pulley and Axle.—Let the masses P lbs. and Q lbs. hang over a pulley. If the pulley and string are very light we can neglect their mass in comparison with the masses P and Q. These masses *should then be large.*



The string should be practically perfectly flexible, so that we can neglect rigidity.

When the mass P *just begins* to fall, the resultant pressure on the bearing, if we neglect mass of rope and pulley, is $Pg + Qg = R$ poundals.

From page 198, Vol. II, *Statics*, we see that for a *new bearing* the friction is given by

$$F = \mu R \frac{\alpha}{\sin \alpha} = \mu g(P + Q) \frac{\alpha}{\sin \alpha} \text{ poundals,}$$

where α is the bearing angle AOB, and μ is the coefficient of static sliding friction. Let a be the radius of pulley and r the radius of axle. Then when P *just begins* to fall we have equilibrium, and the algebraic sum of the moments of the forces, taking rotation counterclockwise positive, is equal to zero. The moment of Q is then $+ Qga$, and of the friction F (page 68) $+ Fr$, since the friction acts opposite to the motion of the axle. The moment of P is $- Pga$, since it acts to cause clockwise rotation.

Therefore, disregarding rigidity of string,

$$Qga + \mu gr(P + Q)\frac{\alpha}{\sin \alpha} - Pga = 0,$$

and we have for the *coefficient of static sliding friction*

$$\mu = \frac{(P - Q)}{P + Q} \cdot \frac{a}{r} \cdot \frac{\sin \alpha}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If P is greater than the mass necessary to just cause motion to begin, let f be the uniform acceleration of P and Q. Then, as in Ex. 13, page 9, the tension of the string on right is $P(g - f)$, and on left $Q(g + f)$. Disregarding the mass of string and pulley, the pressure on the journal is then

$$R = P(g - f) + Q(g + f) = [(P + Q)g - (P - Q)f] \text{ poundals.}$$

The friction *for new bearing* is then, as before,

$$F = \mu R \frac{\alpha}{\sin \alpha} = \frac{\mu \alpha}{\sin \alpha}[(P + Q)g - (P - Q)f] \text{ poundals,}$$

where μ is the coefficient of kinetic axle-friction.

If P falls through the distance s, the work it does is the tension on right multiplied by s, or

$$\text{work of } P = P(g - f)s \text{ ft.-poundals.}$$

Let θ be the angular displacement; then, disregarding rigidity of the string, $a\theta = s$, or $\theta = \frac{s}{a}$. Any point of the journal then passes through the distance $r\theta = \frac{rs}{a}$, where r is the radius of the journal.

The work consumed by friction is then (page 68)

$$-\mu R\frac{\alpha}{\sin\alpha}\cdot\frac{rs}{a} = -\mu\frac{r\alpha}{a\sin\alpha}s[(P+Q)g-(P-Q)f]\text{ ft.-poundals.}$$

The work of raising Q is the tension on left multiplied by s, or

$$-Q(g+f)s\text{ ft.-poundals.}$$

The minus sign is used in both cases because work is done against friction and the tension on left.

Now the work of P must be equal and opposite to the work done against friction and the tension on the left. Hence the algebraic sum must be zero, or

$$P(g-f)s - Q(g+f)s - \mu\frac{r\alpha}{a\sin\alpha}s[(P+Q)g-(P-Q)f] = 0.$$

If we put $f = \frac{2s}{t^2}$, where t is the time of fall, and solve for μ, we obtain

$$\mu = \left[\frac{(P-Q)-(P+Q)\frac{2s}{gt^2}}{(P+Q)-(P-Q)\frac{2s}{gt^2}}\right]\frac{a\sin\alpha}{r\alpha}. \quad . \quad . \quad . \quad (2a)$$

Now P and Q should be made nearly equal in order that the motion may be slow and the space s described in the time t accurately noted. We have also seen that the mass of the pulley and string must be small, and in order that it may be disregarded P and Q should be large.

If these conditions are complied with, $(P-Q)$ will be insignificant compared to $P+Q$, and the second term in the denominator of (2a) can be disregarded. We have then for the coefficient of kinetic sliding friction the practical equation

$$\mu = \left[\frac{P-Q}{P+Q} \div \frac{2s}{gt^2}\right]\frac{a\sin\alpha}{r\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

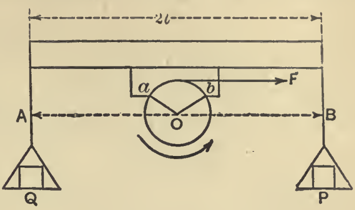
If the bearing angle is small, $\sin\alpha = \alpha$, nearly.

Equation (2b) gives then the coefficient when the masses P and Q are large and nearly equal so that motion is slow, when the mass of pulley and string is small and disregarded, and when the string is practically perfectly flexible so that rigidity is disregarded.

(For the influence of rigidity see Chap. IX, Vol. II, *Statics.*) We have only to observe the distance s described by P in the time t.

Again, we see from (1) and (2b) that the coefficient of kinetic *is less than the coefficient of static sliding friction.*

4. By Friction-brake.—The friction-brake consists of a lever AB of equal arms $AO = BO = l$, so that the entire weight of the lever with scale-pans attached acts upon the centre of the axle O.



If the axle turns, say counter-clockwise, as indicated in the figure, it tends to turn the lever in the same direction. If we put a mass Q in the left pan and a mass P in the right pan and make *P just large enough to keep the lever horizontal*, we have the weights Pg and Qg and the friction F in equilibrium.

If M is the mass of the lever and pans, etc., the pressure on the axle is $(P + Q + M)g = R$.

For a *new bearing* (page 198, Vol. II, *Statics*) the friction is then

$$F = \mu R \frac{\alpha}{\sin \alpha} = \mu g(P + Q + M)\frac{\alpha}{\sin \alpha},$$

where α is the bearing angle aOb and μ is the coefficient of kinetic sliding friction.

The moment of the friction is $-Fr$, where r is the radius of axle. The moment of the weight Qg is $-Qgl$, and of the weight Pg, $+Pgl$, where l is the lever-arm AO or BO. When the lever is just horizontal we have equilibrium and

$$Pgl - Qgl - Fr = 0,$$

or

$$Pgl - Qgl - \mu g(P + Q + M)\frac{r\alpha}{\sin \alpha} = 0,$$

or the coefficient of kinetic sliding friction is

$$\mu = \frac{(P - Q)l \sin \alpha}{(P + Q + M)r\alpha}. \quad \text{. . . . . (4)}$$

Friction-brake Test.—The friction-brake can be used for measuring the work done by an engine when working uniformly. Thus suppose the axle is driven by an engine, and by means of a crank on the axle some machine, as for instance a pump, is worked.

We first count the number of revolutions n per minute while the pump is in action. If then *we disconnect the pump* we shall find the axle to revolve much more rapidly, since the only work now done by the engine is against the friction of the bearing. We now apply the brake and load it at each end until it is horizontal and the axle is slowed up to its former speed of n revolutions per minute. The work done against brake-friction is now equal to the work before consumed by the pump, provided the engine works uniformly.

But the friction is given (page 68) by

$$F = \frac{(P - Q)gl}{r},$$

or in gravitation measure by

$$F = \frac{(P - Q)l}{r}.$$

We have then for the work done in one revolution $2\pi rF$, and in n revolutions per minute the work per minute is $2\pi rnF$. Taking F in gravitation measure or in pounds, and r in feet, this is foot-pounds per minute. If we divide by 33000, we obtain (page 50) horse-power. Hence

$$\text{H.P.} = \frac{\pi rnF}{16500} = \frac{\pi n(P - Q)l}{16500},$$

where F, P and Q are in pounds, l and r in feet, and n is the number of revolutions per minute made while the pump was connected.

Work of Axle-friction. — The friction upon an axle in any case when μ is known is given in Chap. IX, Vol. II, *Statics*. Thus for a *new bearing* we have (page 198, Vol. II, *Statics*)

$$F = \mu R \frac{\alpha}{\sin \alpha},$$

where R is the resultant pressure on the axle and α is the bearing angle. If we substitute this in the place of F in the preceding article, we have the work per minute

$$2\pi rnF = 2\pi\mu Rrn\frac{\alpha}{\sin \alpha},$$

and for the horse-power

$$\text{H.P.} = \frac{\pi\mu Rrn\alpha}{16500 \sin \alpha},$$

where R is taken in pounds, r in feet, n in revolutions per minute. If the bearing angle is small, we have $\alpha = \sin \alpha$ nearly.

Coefficients of Kinetic Sliding Friction. — The following tables give a few values of the value of μ as determined by experiment for kinetic sliding friction and axle-friction.

COEFFICIENTS OF KINETIC SLIDING FRICTION, $\mu = \tan \phi$.

Substances in Contact.		Condition of Surfaces and Kind of Unguent.						
		Dry.	Wet.	Olive Oil.	Lard.	Tallow.	Dry Soap.	Polished and Greasy.
Wood on wood	Minimum......	0.20	....	....	0.06	0.06	0.14	0.08
	Mean..........	0.36	0.25	....	0.07	0.07	0.15	0.12
	Maximum.....	0.48	....	....	0.07	0.08	0.16	0.15
Metal on metal	Minimum......	0.18	....	0.06	0.07	0.07	....	0.11
	Mean..........	0.24	0.31	0.07	0.09	0.09	0.20	0.13
	Maximum.....	0.20	....	0.08	0.11	0.11	....	0.17
Wood on metal	Minimum......	0.20	....	0.05	0.07	0.06	....	0.10
	Mean..........	0.42	0.24	0.06	0.07	0.08	0.20	0.14
	Maximum.....	0.62	....	0.08	0.08	0.10	....	0.16
Hemp ropes or plaits	On wood....	0.45	0.33	....	....	....		
	On iron.....	....	....	0.15	....	0.19		
Leather belts on wood or metal	Raw.......	0.54	0.36	0.16	....	0.20		
	Pounded...	0.30	....	....	....	....		
	Greasy....	....	0.25	....	....	....		
Same on edge for piston-packing	Dry...	0.34	0.31	0.14	....	0.14		
	Greasy	....	0.24	....	....	...		

COEFFICIENTS OF AXLE-FRICTION.

	Dry or Slightly Greasy.	Oil, Tallow, or Lard.		Damp and Greasy.
		Ordinary Lubrication.	Thorough Lubrication.	
Bell-metal on bell-metal	.....	0.097	.....	.....
“ “ “ cast iron	.....	.....	0.049	.....
Wrought iron on bell-metal	0.251	0.075	0.054	0.189
“ “ “ cast iron	.....	0.075	0.054	.....
Cast iron on cast iron	.....	0.075	0.054	0.137
“ “ “ bell-metal	0.194	0.075	0.054	0.161

More extensive tables will be found in treatises on Engineering. Comparing the values in the tables just given with those in the table given on page 192, Vol. II, *Statics*, we see that the coefficient of kinetic *is always less than the coefficient of static sliding friction.*

We see also that for axle-friction in general we have for the coefficient of kinetic friction:

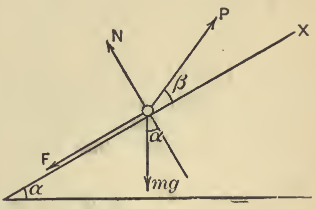
for ordinary lubrication $\mu = 0.070$ to 0.080;

for thorough lubrication $\mu = 0.054$.

EXAMPLES.

(1) *A body of mass m is placed upon the upper side of a rough inclined plane which makes an angle α with the horizontal and is acted upon by a uniform force P which makes the angle β with the plane. Find the friction, the work of friction for any distance described and the motion of the body upon the plane.*

Ans. Consider the body as a particle placed at any point on the plane. We have acting on the particle the weight mg, the force P, the normal reaction N and the friction F, which latter acts always opposed to the direction of motion. Let us take all forces in gravitation measure.



Take OX along the plane upwards and ON away from the plane as the positive directions of X and Y. Then $\theta_x = 90°$, $\theta_y = 0$, and in gravitation measure.

$$R_x' = P \cos \beta - m \sin \alpha,$$
$$R_y' = P \sin \beta - m \cos \alpha,$$

where the angle β is measured from OX counter-clockwise. With this convention, these values of R_x' and R_y' are general. We have then for the normal pressure in general

$$N = -(P \sin \beta - m \cos \alpha) = m \cos \alpha - P \sin \beta,$$

and for the friction

$$F = \mu\,(m \cos \alpha - P \sin \beta)$$

acting always opposite to the direction of motion.

For motion up the plane, we have for the tangential component of the external forces

$$T = P\cos\beta - m\sin\alpha.$$

For motion down the plane

$$T = -P\cos\beta + m\sin\alpha.$$

In general, then,

$$T = \pm P\cos\beta \mp m\sin\alpha,$$

where the upper signs are for motion up and the lower signs for motion down the plane.

The resultant force along the plane is then in all cases

$$T - F = \pm P\cos\beta \mp m\sin\alpha - \mu(m\cos\alpha - P\sin\beta).$$

The acceleration along the plane is then

$$\frac{(T-F)g}{m} = f = g\left[\left(\pm\frac{P}{m}\cos\beta \mp \sin\alpha\right) - \mu\left(\cos\alpha - \frac{P}{m}\sin\beta\right)\right].$$

If in any case f comes out positive, it shows acceleration; if negative, retardation. We see then that f is uniform.

We have then for the space described in any time t (page 51, Vol. I, *Kinematics*), if v_1 is the initial velocity,

$$s = v_1t + \frac{1}{2}ft.$$

For the final velocity,

$$v^2 = v_1^2 + 2fs.$$

For the work done against friction, in gravitation measure,

$$Fs = \mu(m\cos\alpha - P\sin\beta)s.$$

For the work done by or against T, in gravitation measure,

$$Ts = (P\cos\beta - m\sin\alpha)s.$$

For the gain or loss of kinetic energy, in gravitation measure,

$$\frac{1}{2}m(v^2 - v_1^2) = (T - F)s.$$

(2) *A body of* 80 *pounds mass is projected along a rough horizontal plane with a speed of* 50 *ft. per sec. It slides* 155.28 *ft. in coming to rest. Find the coefficient of kinetic sliding friction, the retarding force of friction, and the work done against friction in coming to rest.*

Ans. $\mu = \frac{v^2}{2gs}$, or if $g = 32.2$ ft.-per-sec. per sec, $\mu = 0.25$. Retarding force of friction is 20 lbs.; work done, 3105.6 ft.-lbs.

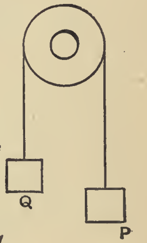
(3) *A body of* 80 *pounds mass is dragged along a rough horizontal plane by means of a mass of* 186 *pounds attached to a string passing over a pulley* (page 8). *It is observed to slide* 10 *feet in the first second, starting from rest. Disregarding rigidity of string and mass and friction of string and pulley, find the coefficient of kinetic sliding friction.* ($g = 32.$)

Ans. $\mu = 0.25$.

(4) *A body placed upon a rough inclined plane whose height is* 1 *ft. and base* 16 *inches is observed to slide* 6.4 *inches in the first second starting from rest. Find the coefficient of kinetic sliding friction.* ($g = 32.$)

Ans. $\mu = 0.25$.

(5) *Two masses* $P = 10$ *lbs. and* $Q = 5$ *lbs. hang by means of a string over a pulley of radius* $a = 6$ *inches. Let the radius of the journal be* $r = 1$ *inch. Let* P *fall from rest a distance* $s = 129.8$ *ft. in a time* $t = 5$ *seconds. Disregarding rigidity of the string and mass of string and pulley, find the coefficient of kinetic axle-friction. Also discuss the action of the apparatus.* ($g = 32\frac{1}{6}$.)



Ans. The tension of string on the left, if f is the acceleration of Q and P, is

$$(Qg + Qf) \text{ poundals.}$$

The tension of string on the right is

$$(Pg - Pf) \text{ poundals.}$$

The pressure on the journal is then, disregarding the mass of string and pulley,

$$R = Q(g + f) + P(g - f) = [(P + Q)g - (P - Q)f] \text{ poundals.}$$

From page 198, Vol. II, *Statics*, we have for a *new bearing* the friction

$$F = \mu R \frac{\alpha}{\sin \alpha} = \frac{\mu \alpha}{\sin \alpha}[(P + Q)g - (P - Q)f] \text{ poundals,}$$

where μ is the coefficient of kinetic friction and α is the angle of the bearing. If α is small, $\alpha = \sin \alpha$ approximately.

If θ is the angular displacement, then, disregarding the rigidity of the string, $a\theta = s$, or $\theta = \frac{s}{a}$. Any point of the journal then passes through the distance $r\theta = \frac{rs}{a}$.

The work consumed by friction then is (page 68)

$$-\mu R \frac{\alpha}{\sin \alpha} \cdot \frac{rs}{a} = -\frac{\mu r \alpha}{a \sin \alpha} \cdot s[(P + Q)g - (P - Q)f] \text{ ft.-poundals.}$$

The work of raising Q is the tension on the left multiplied by s, or

$$-Q(g + f)s \text{ ft.-poundals.}$$

The minus sign is used in both cases because work is done against friction and the left-hand tension.

Now the work of P is the right-hand tension multiplied by s, or

$$P(g - f)s \text{ ft.-poundals.}$$

And this work must be equal and opposite to the work done against friction and the tension on the left. Hence the algebraic sum must be zero, or

$$P(g - f)s - Q(g + f)s - \mu \frac{r\alpha}{a \sin \alpha} s[(P + Q)g - (P - Q)f] = 0. \quad . \quad (1)$$

From (1), if we put $f = \frac{2s}{t^2}$, we have

$$\mu = \left(\frac{(P - Q) - (P + Q)\frac{2s}{t^2}}{(P + Q) - (P - Q)\frac{2s}{t^2}} \right) \frac{a \sin \alpha}{r\alpha};$$

or if α is small, $\alpha = \sin \alpha$ and

$$\mu = \frac{(P-Q)g - (P+Q)\frac{2s}{t^2}}{\frac{r}{a}(P+Q)g - \frac{r}{a}(P-Q)\frac{2s}{t^2}}.$$

Inserting $P = 10$ lbs., $Q = 5$ lbs., $a = \frac{1}{2}$ ft., $r = \frac{1}{12}$ ft., $s = 129.8$ feet, $g = 32\frac{1}{6}$ ft.-per-sec. per sec., we obtain

$$\mu = 0.07.$$

Since $f = \frac{2s}{t^2}$, we have from (1)

$$f = \frac{\left[(P-Q) - \frac{\mu r}{a}(P+Q)\right]g}{(P+Q) - \frac{\mu r}{a}(P-Q)} = 0.323g = 10.39 \text{ ft.-per-sec. per sec.}$$

The velocity at the end of the time t is then

$$v = ft = 51.94 \text{ ft. per sec.}$$

The distance $s = \frac{1}{2}ft^2 = 129.8$ ft., as assumed.

Tension on right $= P(g-f) = 10g - 3.23g = 6.77g$ poundals $= 6.77$ lbs.

Tension on left $= Q(g+f) = 5g + 1.615g = 6.615g$ poundals $= 6.615$ lbs.

The work of friction is

$$-F\frac{rs}{a} = -\frac{\mu r}{a}s[P+Q)g - (P-Q)f] = 20.28g \text{ ft.-pdls.} = 20.28 \text{ ft.-lbs.}$$

Again, the work of $P = 6.77 \times 129.8 = 879.136$ ft.-lbs.;
the work of $Q = 6.615 \times 129.8 = 858.855$ ft.-lbs.

The difference of these works $= 20.28$ ft.-lbs. $=$ work of friction.

The *power* of P (page 49) $= \frac{879.136}{5} = 175.827$ ft.-lbs. per sec., or

$$\frac{175.827}{550} = 0.319 \text{ horse-power.}$$

The rate of work of the "useful" resistance $= \frac{858.855}{5} = 171.771$ ft.-lbs. per sec. Hence the *efficiency* of the machine (page 52) is

$$\epsilon = \frac{171.77}{175.83} = 0.97.$$

The efficiency in general is

$$\epsilon = \frac{Q(g+f)s}{P(g-f)s} = \frac{1 - \frac{\mu r}{a}}{1 + \frac{\mu r}{a}}.$$

(6) *In the preceding example, what mass P will raise $Q = 5$ lbs. a distance of 20 feet in 3 seconds?*

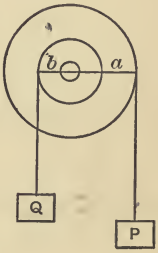
Ans. $f = \dfrac{\left[(P-Q) - \dfrac{\mu r}{a}(P+Q)\right]g}{(P+Q) - \dfrac{\mu r}{a}(P-Q)}$. Therefore we have

$$P = \frac{Q(g+f) + \dfrac{\mu r}{a}Q(g+f)}{(g-f) - \dfrac{\mu r}{a}(g-f)}.$$

Substituting $f = \dfrac{2s}{t^2} = \dfrac{40}{9}$, we have $P = 6.759$ lbs.

(7) *In a wheel and axle the radius of the wheel is $a = 3$ ft., of the axle $b = 2$ ft. Let $r = 1$ inch be the radius of the journal, and $\mu = 0.07$ be the coefficient of kinetic friction. Let the moving mass $P = 10$ lbs. and the mass lifted be $Q = 5$ lbs. Let P start from rest and fall for a time $t = 5$ seconds. Disregarding rigidity of the string and the mass of string and wheel and axle, discuss the apparatus.* ($g = 32\frac{1}{6}$.)

Ans. Let f be the acceleration of P. Then $\dfrac{b}{a}f$ will be the acceleration of Q. Also if P falls the distance s, Q rises the distance $\dfrac{b}{a}s$. We have then, just as in example (5),



Tension on left $= Q\left(g + \dfrac{b}{a}f\right)$ poundals.

Tension on right $= P(g - f)$ poundals.

Pressure on the journal

$$\left[R = (P+Q)g - \left(P - Q\frac{b}{a}\right)f\right] \text{ poundals.}$$

The friction for *new bearing* (page 198, Vol. II, *Statics*) is

$$F = \frac{\mu\alpha}{\sin\alpha}\left[(P+Q)g - \left(P - Q\frac{b}{a}\right)f\right] \text{ poundals,}$$

where μ is the coefficient of kinetic friction and α is the angle of bearing.

The work consumed by friction is

$$-F\frac{rs}{a} = -\frac{\mu r\alpha}{a\sin\alpha}s\left[(P+Q)g - \left(P - Q\frac{b}{a}\right)f\right] \text{ ft.-poundals.}$$

The work of raising Q is the tension on left multiplied by $\dfrac{b}{a}s$, or

$$-Q\frac{b}{a}s\left(g + \frac{b}{a}f\right) \text{ ft.-poundals.}$$

The minus sign is used in both cases because work is done against friction and the left-hand tension.

The work of P is the right-hand tension multiplied by s, or

$$P(g-f)s \text{ ft.-poundals,}$$

and this work must be equal and opposite to the work done against friction and the left-hand tension. Hence the algebraic sum must be zero, or

$$P(g-f)s - Q\frac{b}{a}s\left(g+\frac{b}{a}f\right) - \frac{\mu r\alpha}{a\sin\alpha}s\left[(P+Q)g - \left(P-Q\frac{b}{a}\right)f\right] = 0. \qquad (1)$$

From (1) we obtain for the acceleration of P

$$f = \frac{\left(P-Q\frac{b}{a}\right)g - \frac{\mu r\alpha}{a\sin\alpha}(P+Q)g}{\left(P+Q\frac{b^2}{a^2}\right) - \frac{\mu r\alpha}{a\sin\alpha}\left(P-Q\frac{b}{a}\right)}.$$

If α is small, $\sin\alpha = \alpha$ and

$$f = \frac{\left(P-Q\frac{b}{a}\right)g - \frac{\mu r}{a}(P+Q)g}{\left(P+Q\frac{b^2}{a^2}\right) - \frac{\mu r}{a}\left(P-Q\frac{b}{a}\right)} = 0.544g = 17.49 \text{ ft.-per-sec. per sec.}$$

The acceleration of Q is then

$$\frac{b}{a}f = 0.363g = 11.66 \text{ ft.-per-sec. per sec.}$$

The velocity of P at the end of the time $t = 5$ sec. is

$$v = ft = 87.44 \text{ ft. per sec.},$$

and the velocity of Q is

$$\frac{b}{a}v = 58.29 \text{ ft. per sec.}$$

The distance s passed through by P is

$$s = \frac{1}{2}ft^2 = 218.59 \text{ ft.},$$

and the distance passed through by Q is

$$\frac{b}{a}s = 145.73 \text{ ft.}$$

Tension on right $= P(g-f) = 10g - 5.44g = 4.56g$ poundals $= 4.56$ lbs.

Tension on left $= Q\left(g+\frac{b}{a}f\right) = 5g - 1.817g = 6.82g$ poundals $= 6.82$ lbs.

The work consumed by friction $= 4.1g$ ft.-poundals $= 4.1$ ft.-pounds.

The work of tension on right $= 997.56g$ ft.-poundals $= 997.56$ ft.-lbs.

The work of tension on left $= 993.46g$ ft.-poundals $= 993.46$ ft.-lbs.

The difference of these works $= 4.1$ ft.-lbs. $=$ work of friction.

The *power* of P (page 49) $= \dfrac{997.56}{5} = 199.51$ ft.-lbs. per sec., or

$$\frac{199.51}{550} = 0.363 \text{ horse-power.}$$

The *efficiency* of the machine (page 52) is

$$\epsilon = \frac{993.46}{997.56} = 0.996.$$

The efficiency in general is

$$\epsilon = \frac{Q\left(g+\frac{b}{a}f\right)\frac{b}{a}s}{P(g-f)s} = \frac{1-\frac{\mu r\alpha}{a\sin\alpha}}{1+\frac{\mu r\alpha}{b\sin\alpha}}.$$

(8) *In the preceding example what mass P will raise $Q = 5$ lbs. a distance of 20 ft. in 3 seconds?*

Ans. $s = 20$, $t = 3$, $f = \frac{2s}{t^2} = \frac{40}{9}$. Hence

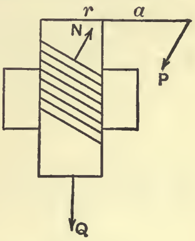
$$P = \frac{Q\frac{b}{a}\left(g+\frac{b}{a}f\right) + \frac{\mu r}{a}Q\left(g+\frac{b}{a}f\right)}{(g-f) - \frac{\mu r}{a}(g-f)} = 4.244g \text{ poundals} = 4.244 \text{ lbs.}$$

(9) *A friction-brake of $M = 15$ lbs. mass is balanced on a rotating shaft of radius $r = 6$ inches, by masses of $Q = 10$ lbs. and $P = 10$ lbs. 10 oz. Find the coefficient of kinetic friction and the friction. Also if the shaft makes 60 revolutions per minute find the rate of work of the friction.*

Ans. $\mu = 0.07$, $F = 2.5$ lbs. Rate of work of friction $= 7.854$ ft.-lbs. per sec., or 0.01428 horse-power.

(10) *A screw of radius $r = 1$ inch is acted upon by a force oj $P = \frac{1}{2}$ lb. with a constant lever-arm of $a = 1$ ft. and overcomes a resistance of $Q = 5$ lbs. If the angle of the thread is $\alpha = 45°$ find the coefficient of kinetic sliding friction if the number of revolutions per minute is 60. Also find the efficiency, and the acceleration of P. Disregard the mass of the screw, and take $g = 32\frac{1}{6}$ ft.-per-sec. per sec.*

Ans. Let P be the force applied at the end of the arm a, and let the radius of the screw be r, the pitch p, and the resistance Q.



If N is the sum of the normal pressures and α the inclination of the thread to the horizontal, we have $N = \frac{Q}{\cos\alpha}$, and the friction $F = \mu N = \frac{\mu Q}{\cos\alpha}$, where μ is the coefficient of friction.

Let f be the acceleration of P. Then the moving force is $P(g-f)$ poundals. If s is the distance passed through by P in any time t, then the work of the moving force is

$$P(g-f)s \text{ ft.-poundals.}$$

The resistance Q is overcome through the distance $\frac{p'}{2\pi a}s$. The work of overcoming the resistance is then

$$-\frac{Qg}{a}\cdot\frac{p}{2\pi}s \text{ ft.-poundals.}$$

The friction is overcome through the distance $\frac{r}{a}\cdot\frac{s}{\cos\alpha}$. The work of overcoming the friction is then

$$-\frac{\mu Qg}{\cos\alpha}\cdot\frac{rs}{a\cos\alpha}.$$

The minus sign is used because work is done against friction and the resistance.

The work of $P(g-f)$ must be equal and opposite to the work done against friction and the resistance. Hence the algebraic sum must be zero, or

$$P(g-f)s - \frac{Qgps}{2\pi a} - \frac{\mu Qgrs}{a \cos^2 \alpha} = 0.$$

From this we have, since $\frac{p}{2\pi r} = \tan \alpha$ and $f = \frac{2s}{t^2}$,

$$f = g\left[1 - \frac{Qr}{Pa \cos^2 \alpha}(\sin \alpha \cos \alpha + \mu)\right] = \frac{2s}{t^2}, \quad . \quad . \quad . \quad . \quad (1)$$

and from (1), for the coefficient of kinetic friction,

$$\mu = \frac{Pa}{Qr}\cos^2 \alpha - \sin \alpha \cos \alpha\left(1 + \frac{2sPa}{gt^2 Qr \tan \alpha}\right). \quad . \quad . \quad . \quad . \quad (2)$$

For the efficiency we have

$$\epsilon = \frac{\frac{Qgps}{2\pi a}}{\frac{Qgps}{2\pi a} + \frac{\mu Qgrs}{a \cos^2 \alpha}} = \frac{1}{1 + \frac{\mu}{\sin \alpha \cos \alpha}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $f = 0$ *we have equilibrium*, and from (1) we have in this case

$$P = \frac{Qr}{a}\left(\tan \alpha + \frac{\mu}{\cos^2 \alpha}\right),$$

or the same as already found, Ex. (11), page 219, Vol. II, *Statics*.

In this case (2) becomes the coefficient of static friction

$$\mu = \frac{Pa}{Qr}\cos^2 \alpha - \sin \alpha \cos \alpha.$$

We see from (3) that the efficiency is a maximum when $\sin \alpha \cos \alpha$ is a maximum, or when $\sin \alpha = \cos \alpha$ or $\alpha = 45°$.

If n is the number of revolutions per minute, the distance s described in one minute is $2\pi an$. We have then

$$\frac{2s}{gt^2} = \frac{4\pi an}{60 \times 60 \times g} = \frac{\pi an}{900g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Inserting in these equations the values $a = 1$ ft., $r = \frac{1}{12}$ ft., $P = \frac{1}{2}$ lb., $Q = 5$ lbs., $\alpha = 45°$, $n = 60$, $g = 32\frac{1}{6}$ ft.-per-sec. per sec., we have

$$\mu = 0.096, \quad \epsilon = 0.84, \quad f = 0.007, \quad g = 0.225 \text{ ft.-per-sec. per sec.}$$

(11) *A train runs on a horizontal track with the speed v_1, and by the application of brakes to the driving-wheels of the locomotive the speed is reduced to the speed v. Find the distance and time of running during the reduction of speed, disregarding all resistances other than those due to the action of the brakes.*

Ans. Let m be the mass of the train in pounds, v_1 the initial and v the final speed in feet per second, s the distance in feet, and t the corresponding time in seconds.

Let n be the number of driving-wheels braked, P the pressure of each brake, and R the pressure of each braked wheel on the rails.

Let μ_r be the coefficient of kinetic sliding friction for the wheels and *rails*, and μ_b the coefficient of kinetic sliding friction for the wheels and *brakes*.

We may distinguish three cases:

1st. *The wheels roll without slipping on the rails.*—In this case we have the friction $n\mu_b Pg$ on the brakes less than the friction $n\mu_r Rg$ of sliding on the rails, or

$$\mu_b P < \mu_r R. \qquad (1)$$

When condition (1) is satisfied the work of stoppage is due to the work of friction of the brakes, and we have the change of kinetic energy equal to the work done against friction, or

$$\frac{1}{2}m(v^2 - v_1^2) = - n\mu_b Pgs.$$

Hence

$$s = \frac{m(v_1^2 - v^2)}{2n\mu_b Pg}, \qquad (2)$$

and

$$t = \frac{2s}{v_1 + v} = \frac{m(v_1 - v)}{n\mu_b Pg}. \qquad (3)$$

If the train is brought to rest we have $v = 0$ in these equations.

2d. *The wheels just on the point of slipping on the rails.*—In this case we have the friction $n\mu_b Pg$ on the brakes just equal to the friction $n\mu_r Rg$ of sliding on the rails, or

$$\mu_b P = \mu_r R. \qquad (4)$$

When condition (4) is satisfied the work of stoppage is due to the work of friction of the brakes, and we have

$$\frac{1}{2}m(v^2 - v_1^2) = - n\mu_b Pgs = - n\mu_r Rs.$$

Hence

$$s = \frac{m(v_1^2 - v^2)}{2n\mu_b Pg} = \frac{m(v_1^2 - v^2)}{2n\mu_r Rg}; \qquad (5)$$

$$t = \frac{m(v_1 - v)}{n\mu_b Pg} = \frac{m(v_1 - v)}{n\mu_r Rg}. \qquad (6)$$

If the train is brought to rest we have $v = 0$ in these equations.

3d. *The wheels slip on the rails.*—In this case we have the friction $n\mu_b Pg$ on the brakes greater than the friction $n\mu_r Rg$ of sliding on the rails, or

$$\mu_b P > \mu_r R. \qquad (7)$$

When condition (7) is satisfied the work of stoppage is due to the work of friction of the rails, and we have

$$\frac{1}{2}m(v^2 - v_1^2) = - n\mu_r Rgs.$$

Hence

$$s = \frac{m(v_1^2 - v^2)}{2n\mu_r Rg}; \qquad (8)$$

$$t = \frac{m(v_1 - v)}{n\mu_r Rg}. \qquad (9)$$

If the train is brought to rest we have $v = 0$ in these equations.

We see by inspection that the distance s given by (5) is less than the distance s given by (2) or (8).

That is, *the least distance and time of stoppage is when the wheels just roll without sliding*, or when

$$P = \frac{\mu_r}{\mu_b} R.$$

If we take the coefficients for wheels and brakes and wheels and rails equal we have $\mu_b = \mu_r$, and the conditions for the three cases are respectively

$$P < R, \quad P = R, \quad P > R.$$

The distance and time are then the least possible when $P = R$. If P is greater than R, the wheels slip and the distance and time are greater than when $P = R$.

(12) *In the preceding example suppose the grade rises h feet in a length of l feet and base of b feet.*

Ans. 1st. *Wheels roll without slipping.*—In this case we have

$$\mu_b P < \mu_r \frac{b}{l} R; \quad \ldots\ldots \quad (1)$$

$$s = \frac{m(v_1^2 - v^2)}{2g\left(n\mu_b P \pm \frac{mh}{l}\right)}; \quad \ldots\ldots \quad (2)$$

$$t = \frac{m(v_1 - v)}{g\left(n\mu_b P \pm \frac{mh}{l}\right)}; \quad \ldots\ldots \quad (3)$$

where the plus sign is for train running up grade, and the minus sign for train running down grade.

2d. *Wheels just on the point of slipping.*—In this case we have

$$\mu_b P = \mu_r \frac{b}{l} R; \quad \ldots\ldots \quad (4)$$

$$s = \frac{m(v_1^2 - v^2)}{2g\left(n\mu_b P \pm \frac{mh}{l}\right)} = \frac{m(v_1^2 - v^2)}{2g\left(n\mu_r \frac{b}{l} R \pm \frac{mh}{l}\right)}; \quad \ldots\ldots \quad (5)$$

$$t = \frac{m(v_1 - v)}{g\left(n\mu_b P \pm \frac{mh}{l}\right)} = \frac{m(v_1 - v)}{g\left(n\mu_r \frac{b}{l} R \pm \frac{mh}{l}\right)}; \quad \ldots\ldots \quad (6)$$

where the plus sign is for train running up grade, and the minus sign for train running down grade.

3d. *Wheels slip on the rails.*—In this case we have

$$\mu_b P > \mu_r \frac{b}{l} R; \quad \ldots\ldots \quad (7)$$

$$s = \frac{m(v_1^2 - v^2)}{2g\left(n\mu_r \frac{b}{l} R \pm \frac{mh}{l}\right)}; \quad \ldots\ldots \quad (8)$$

$$t = \frac{m(v_1 - v)}{g\left(n\mu_r \frac{b}{l} R \pm \frac{mh}{l}\right)}; \quad \ldots\ldots \quad (9)$$

where the plus sign is for train running up grade, and the minus sign for train running down grade.

(13) *A train is running at the rate of* 45 *miles per hour. If the brakes press with two thirds the weight on the wheels of the locomotive and if the locomotive is one half the weight of the train and the coefficient of kinetic sliding friction is* 0.18, *find the distance and time in coming to rest on a horizontal track.* $(g = 32.)$

Ans. $s = 378\frac{1}{8}$ ft., disregarding all resistances except that due to the brakes.
$t = 11.46$ sec.

(14) *A train is running at the rate of* 60 *miles per hour on a horizontal track. If the brake pressure is two thirds of the weight of the train, the coefficient of kinetic sliding friction* 0.18, *and the train resistance* 20 *lbs. per ton, find the distance and time in coming to rest.* $(g = 32.)$

Ans. $s = 938.5$ ft.; $t = 21.33$ sec.

(15) *If a force of* 20 *lbs. per ton of load is required to maintain the speed of a train on a level track, find the coefficient of sliding friction between the driving-wheels and rails when a locomotive of* 27 *tons can just maintain the speed of a train of* 252 *tons.*

Ans. $\mu = \frac{1}{12}$.

(16) *A weight of* 10 *tons is dragged in half an hour* 330 *feet up a plane inclined* 30° *to the horizontal, the coefficient of kinetic sliding friction being* $\frac{1}{\sqrt{3}}$. *Find the work expended and the horse-power of an engine by which the work could be done.*

Ans. 7392000 foot-pounds; $7\frac{7}{15}$ horse-power.

(17) *An inclined plane is partly smooth and partly rough* $\left(\mu = \frac{\sqrt{3}}{2}\right)$; *a particle slips down the upper smooth part and moves on to the rough part; the inclination of the plane is* 30° *and the length of the smooth part is* 4 *feet. Find the distance described before it comes to rest.*

Ans. 8 ft. on the rough part.

(18) *If the height of an inclined plane is* 12 *feet, the base* 16 *feet, find how far a body will move on the horizontal plane, supposing it to pass from one plane to the other without loss of velocity, the coefficient for both planes being* 1/8.

Ans. 80 feet.

(19) *A heavy slab whose under surface is rough, but the upper smooth, slides down an inclined plane. Find the acceleration with which a particle on its upper surface will move along the slab if the angle of inclination of the plane is* α *and the coefficient* μ, *the mass of the slab* M, *and of the particle* m.

Ans. $\mu\frac{M+m}{M}g\cos\alpha$.

CHAPTER VII.

CONSERVATION OF ENERGY—LAW OF ENERGY.

Conservative Forces. — Forces which depend solely upon the *position* of a particle are called **conservative forces**, because, as we shall see presently, the principle of conservation of energy holds good when such forces only exist.

Non-conservative Forces.—Forces which do not depend solely upon the position of a particle are called **non-conservative forces**, because for such forces the principle of conservation of energy no longer holds.

The force of gravity upon a particle depends solely upon the position of the particle and is therefore a conservative force. So is the elastic force of a spring which depends solely upon configuration, and so also are the forces of nature generally. But an applied force which is independent of position is non-conservative. The resisting force of friction, and in general all resistances to motion, do not depend solely upon position and are therefore non-conservative.

Potential Energy.—The work which a body is capable of doing *by reason of its position, under the action of its conservative forces only*, is called its **potential energy**.

Thus a mass which is suspended at a distance above the earth can do work when released. A bent spring can do work when released, etc.

This form of energy is sometimes called energy of position or *static energy*, to distinguish it from *kinetic energy* (page 56), and to denote its independence of velocity.

Conservation of Energy.—If the forces F_1, F_2, F_3, etc., act upon a particle of mass m, and s_1, s_2, s_3, etc., are the indefinitely small displacements in the direction of the forces, then, since the forces may be taken as uniform during the indefinitely small displacements, we have for the total work of the forces

$$F_1s_1 + F_2s_2 + F_3s_3 + \dots = \Sigma Fs.$$

If the velocity of the mass m is increased from v_1 to v, we have (page 57) the gain of kinetic energy equal to the work done by the forces, or

$$\frac{1}{2}m(v^2 - v_1^2) = \Sigma Fs.$$

Now if the forces depend solely upon the position of the particle, the work ΣFs is the decrease of potential energy, taking potential energy as just defined.

Hence, *the gain of kinetic energy is equal to the loss of potential energy, and vice versa.*

Let the sum of the kinetic and potential energy at any instant be the **total energy** at that instant. If then E_1 is the total initial energy and E is the total final energy, we must have *the total energy before equal to the total energy after displacement*, or

$$E = E_1, \quad \text{or} \quad E - E_1 = 0. \qquad \dots\dots \quad (1)$$

That is, *the total gain or loss of energy when all the forces depend solely upon position is zero.*

This is called the principle of **conservation of energy**. Since it holds only when all the forces depend solely upon position, we have called such forces **conservative forces**. We have therefore defined potential energy as the work which a body is capable of doing by reason of its position under the action of its conservative forces only.

Law of Energy.—The principle of conservative of energy just stated holds good then only when all the forces are conservative or depend solely upon position.

Such forces, as already stated (page 86), are the force of gravity, the force of elasticity, and in general all the forces of nature.

But for the resistance of friction, for resistances generally, and for all applied forces which do not depend solely upon position, the total gain or loss of energy must evidently be equal to the work done by or against these forces.

Let the sum of the kinetic and potential energy at any instant be the total energy at that instant.

If then E_1 is the total initial and E is the total final energy, and ΣFs is the algebraic sum of the works of those applied forces which do not depend solely upon position, taking work positive (+) when a force is in the direction of displacement and negative (−) when it is opposite to the direction of displacement, we have

$$E - E_1 = \Sigma Fs. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Since then the principle of conservation of energy does not hold for forces which do not depend solely upon position, such forces are called **non-conservative**.

Hence, *the total gain or loss of energy is equal to the work done by or against non-conservative forces.*

This is called the **law of energy**, and the principle of conservation of energy is evidently only a special case, when there are no non-conservative forces, and $\Sigma Fs = 0$.

If the non-conservative forces are uniform they do not change with the displacement, and

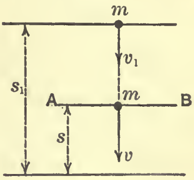
$$\Sigma Fs = F_1s_1 + F_2s_2 + F_3s_3 + \ldots$$

may be taken for any displacement large or small.

If the non-conservative forces are not uniform we must take ΣFs for an indefinitely small displacement.

Application of the Law of Energy to Kinetic Problems.—The law of energy is a generalized form of the laws of motion and may be applied directly to the solution of kinetic problems.

1. *Motion of a Falling Body—no Resistances.*—Let a particle of mass m be acted upon by gravity only and have the initial velocity v_1 at the distance s_1 from the earth.



Under the action of gravity alone the particle falls through the distance $(s_1 - s)$ and acquires the velocity v at the distance s from the earth.

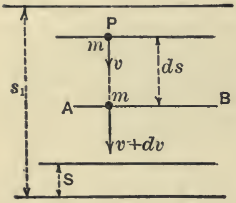
If we consider the force of gravity mg as a force depending solely upon the position of the particle, which it really is, it is a *conservative* force. If we disregard the slight change in g for ordinary distances, the initial potential energy is $+ mgs_1$ and the final potential energy is $+mgs$. The initial and final kinetic energy is $\frac{1}{2}mv_1^2$ and $\frac{1}{2}mv^2$. We have then the initial total energy $E_1 = \frac{1}{2}mv_1^2 + mgs_1$ and the final total energy $E = \frac{1}{2}mv^2 + mgs$. By the principle of the conservation of energy, then.

$$E - E_1 = 0, \quad \text{or} \quad \frac{1}{2}mv^2 + mgs - \frac{1}{2}mv_1^2 - mgs_1 = 0,$$

or

$$v^2 = v_1^2 - 2g(s - s_1).$$

This is the same result as on page 93, Vol. I, *Kinematics*.



[If, however, we do not disregard the slight change in g for ordinary distances, take the plane AB at an indefinitely small distance ds below any point P. Let v be the velocity at the point P, then $v + dv$ is the velocity at the plane AB. The initial potential energy is then $+ mgds$ with reference to the plane AB, and the final potential energy is zero with reference to this plane. The initial and final kinetic energy is $\frac{1}{2}mv^2$ and $\frac{1}{2}m(v + dv)^2$. We have then the initial total energy

$$E_1 = \frac{1}{2}mv^2 + mgds$$

and the final total energy $\frac{1}{2}m(v + dv)^2$.

By the principle of conservation of energy, then, $E - E_1 = 0$, or

$$\frac{1}{2}m(v + dv)^2 - \frac{1}{2}mv^2 - mgds = 0.$$

Expanding and disregarding dv^2, we have

$$vdv = gds.$$

Integrating, we obtain

$$v^2 = 2gs + \text{Const.}$$

Let $v = v_1$ when $s + s_1$. Then Const. $= v_1^2 - 2gs$, and we have, as before,

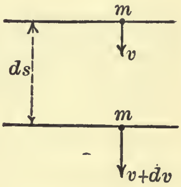
$$v^2 = v_1^2 - 2g(s - s_1).$$

Again, we may regard the force of gravity mg, since for ordinary distances it is practically constant, as a force independent of position, and therefore *non-conservative.*

In this case there is no initial potential energy, since by definition (page 86) potential energy is the work which a body is capable of doing by reason of its position, *under the action of its conservative forces only.* The total initial energy is then $E_1 = \frac{1}{2}mv_1^2$, or entirely kinetic. The total final energy is $E = \frac{1}{2}mv^2$, or also entirely kinetic. The work by non-conservative forces is $+ mg(s_1 - s)$, since the force mg acts in the direction of the motion, and by the law of energy $E - E_1 = + mg(s_1 - s)$, or the total gain of energy is equal to the work done by non-conservative forces, or as before

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = + mg(s_1 - s), \quad \text{or} \quad v^2 = v_1^2 - 2g(s - s_1).$$

[(2) *Motion of a Falling Body—Resistance of Air included.*—Let the particle encounter a resistance to its motion, due to the resistance of the air, which varies as the square of its velocity. Let us denote it by mcv^2, where c is the coefficient of resistance (page 61). This is a non-conservative force, and acts opposite to the motion of the particle. Its work is then $- mcv^2ds$.



Then as before, if we treat mg as a conservative force, the total gain of energy is

$$E - E_1 = \frac{1}{2}m(v + dv)^2 - \left(\frac{1}{2}mv^2 + mgds\right);$$

and since by the law of energy the total loss of energy is equal to the work done by non-conservative forces, we have

$$\frac{1}{2}m(v + dv)^2 - \left(\frac{1}{2}mv^2 + mgds\right) = - mcv^2ds.$$

Expanding and neglecting dv^2,

$$vdv - gds = - cv^2ds.$$

This gives us

$$\frac{vdv}{ds} = g - cv^2,$$

or, since $ds = vdt$,

$$\frac{dv}{dt} = g - cv^2.$$

This is equation (1) of page 111, Vol. I, *Kinematics*.

Again, if we treat mg as a non-conservative force also, there is no potential energy since there are no conservative forces. We have then the initial energy $E_1 = \frac{1}{2}mv^2$, the final energy $E = \frac{1}{2}m(v + dv)^2$, and the algebraic sum of the works of the non-conservative forces is $+ mgds - mcv^2ds$. Hence by the law of energy

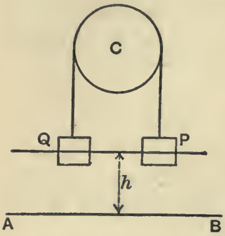
$$E - E_1 = \Sigma Fs,$$

or

$$\frac{1}{2}m(v + dv)^2 - \frac{1}{2}mv^2 = mgds - mcv^2ds.$$

We thus have the same result as before.]

(3) *Let two masses P and Q hang by a perfectly flexible inextensible string over a pulley.*—Let P be the larger mass, and disregard friction and the mass of the pulley and string.



We have then only the conservative forces of gravity, and by the law of energy

$$E - E_1 = 0,$$

or the gain of energy is zero.

Take the centres of mass of P and Q at the same distance h from the plane AB, and let the masses start from rest. The initial energy is then, if we disregard the change of g for small distances, $E_1 = Pgh + Qgh$, or all potential.

At the end of the time t suppose that P has fallen and Q risen a distance s, while both masses have the velocity v, one down and the other, Q, up. Then the final energy of P is $Pg(h - s) + \frac{P}{2}v^2$, the first term potential and the second kinetic energy. The final energy of Q is in like manner $Qg(h + s) + \frac{Q}{2}v^2$. The total final energy is then

$$E = Pg(h - s) + \frac{P}{2}v^2 + Qg(h + s) + \frac{Q}{2}v^2.$$

Since the gain of energy is zero when there are no non-conservative forces, we have $E - E_1 = 0$, or $E_1 = E$, or

$$Pgh + Qgh = Pg(h - s) + \frac{P}{2}v^2 + Qg(h + s) + \frac{Q}{2}v^2,$$

or

$$Pgs - Qgs = \frac{P}{2}v^2 + \frac{Q}{2}v^2.$$

We see from this equation that the loss of potential energy *of the system*, or $Pgs - Qgs$, equals the gain of kinetic energy *of the system*, or $\frac{P}{2}v^2 + \frac{Q}{2}v^2$. Also the loss of potential energy of P, or Pgs, equals the gain of potential energy of Q, or Qgs, plus the gain of kinetic energy of the system.

If we substitute $s = \frac{1}{2}ft^2$, $v = ft$, where f is the constant acceleration for each mass (page 51, Vol. I, *Kinematics*), we obtain

$$f = \frac{(P - Q)g}{P + Q}.$$

This is the same result as on page 9, Ex. 13, or page 53, Ex. 1.

Again, if we consider the weights Pg and Qg as constant forces not depending upon position and therefore non-conservative, there is no potential energy; and since the masses start from rest there is no initial kinetic energy. The initial energy E_1 is then zero, while the final energy is $E = \frac{P}{2}v^2 + \frac{Q}{2}v^2$. The algebraic sum of the works of the non-conservative forces is $+ Pgs - Qgs$. We have then by the law of energy directly, $E - E_1 = \Sigma Fs$, or

$$\frac{P}{2}v^2 + \frac{Q}{2}v^2 = Pgs - Qgs.$$

That is, the gain of energy *of the system* is equal to the algebraic sum of the works of the non-conservative forces.

(4) *In the preceding example take the friction of the axle into account.*—Let the friction on the axle as found page 77, Ex. 5, be

$$\frac{\mu\alpha}{\sin\alpha}[P(g-f) + Q(g+f)],$$

where μ is the coefficient of kinetic friction and α is the bearing angle.

Then if a is the radius of the pulley and r the radius of the journal, and s is the distance through which P falls, the work of friction is, since the friction is opposite to the motion,

$$-\mu\frac{r\alpha}{a\sin\alpha}s[P(g-f) + Q(g+f)].$$

Then, since the gain of energy equals the work of non-conservative forces, we have, since as before disregarding the change of g for small distances, $E_1 = Pgh + Qgh$, and

$$E = Pg(h-s) + \frac{P}{2}v^2 + Qg(h+s) + \frac{Q}{2}v^2;$$

$$Pg(h-s) + \frac{P}{2}v^2 + Qg(h+s) + \frac{Q}{2}v^2 - Pgh - Qgh$$
$$= -\mu\frac{r\alpha}{a\sin\alpha}s[(gP-f) + Q(g+f).$$

Reducing this, we have

$$Pgs - Qgs = \frac{P}{2}v^2 + \frac{Q}{2}v^2 + \mu\frac{r\alpha}{a\sin\alpha}s[P(g-f) + Q(g+f)].$$

We see from this equation that the loss of potential energy of the system, or $Pgs - Qgs$, equals the gain of kinetic energy of the system, or $\frac{P}{2}v^2 + \frac{Q}{2}v^2$, plus the work of overcoming the friction. Also, the loss of potential energy of P, or Pgs, equals the gain of potential energy of Q, or Qgs, plus the gain of kinetic energy of the system, plus the work of overcoming the friction.

If we substitute $s = \frac{1}{2}ft^2$, $v = ft$, where f is the constant acceleration for each mass (page 51, Vol. I, *Kinematics*) we have, when α is small so that $\alpha = \sin\alpha$, approximately,

$$f = \frac{(P-Q)g - \mu\frac{r}{a}(P+Q)g}{P + Q - \mu\frac{r}{a}(P-Q)}.$$

This is the same result as on page 78, Ex. 5.

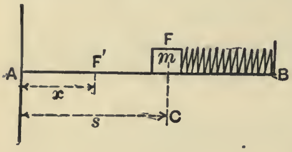
Again, if we treat Pg and Qg as constant forces not depending upon position and therefore non-conservative, there is no potential energy, and the initial energy is $E_1 = 0$, while the final energy is $E = \frac{P}{2}v^2 + \frac{Q}{2}v^2$. The algebraic sum of the works of the non-conservative forces is

$$\Sigma Fs = + Pgs - Qgs - \mu\frac{r}{a}s[P(g-f) + Q(g+f)].$$

We have then directly $E - E_1 = \Sigma Fs$, or

$$\frac{P}{2}v^2 + \frac{Q}{2}v^2 = Pgs - Qgs - \mu\frac{r}{a}s[P(g-f) + Q(g+f)].$$

(5) *Let a spring whose unstrained length is AB be fixed at the end B and compressed from A to C, where it presses against a body of mass m perfectly free to move. Disregarding the mass of the spring, find the motion.*—Let the force at any distance x from A be F', and at the distance $AC = s$ be F. Then we have



$$F' : x :: F : s, \quad \text{or} \quad F' = F\frac{x}{s}.$$

The initial energy is all potential, or $E_1 = Fs$. The final energy at any point x is $E = \frac{1}{2}mv^2 + F'x$. Since there are no non-conservative forces, we have

$$E - E_1 = 0, \quad \text{or} \quad \frac{1}{2}mv^2 + F'x - Fs = 0.$$

Inserting $F'x = F\frac{x^2}{s}$, we have

$$v^2 = \frac{2F}{ms}(s^2 - x^2).$$

This is the same result as obtained page 11, Ex. 22.

Application of the Law of Energy to Static Problems.—The law of energy may also be employed in the solution of problems of equilibrium.

A particle in equilibrium must either be at rest or it must move with uniform velocity. In an indefinitely small displacement, then, whether actual or virtual, there can be no change of kinetic energy. The only possible change of energy, then, is change of potential energy, and the law of energy (page 87) in this case becomes

The gain or loss of potential energy for any indefinitely small displacement is equal to the work done by or against non-conservative forces.

This is the same as the principle of virtual work (page 160, Vol. II, *Statics*). If the non-conservative forces are uniform they do not change with the displacement, and the principle then holds good for any displacement, large or small.

If P_1 is the initial and P the final potential energy, we then have

$$P - P_1 = \Sigma Fs,$$

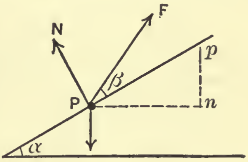
where ΣFs is the algebraic sum of the works of the non-conservative forces, taking work positive (+) when a force is in the direction of the displacement and negative (−) when the force is opposite to the displacement.

If we have conservative forces only, we have then

$$P - P_1 = 0;$$

or, *for any indefinitely small displacement the potential energy must be constant, for equilibrium.*

(1) *Equilibrium of a Particle on an Inclined Plane.*—Let a particle of mass m acted upon by a force F be in equilibrium on a smooth inclined plane.



Let α be the angle of inclination of the plane, and β the angle of F with the plane.

In order to find F, suppose a virtual displacement $Pp = d$ at right angles to the normal reaction N of the plane. Then the work of N during displacement is zero.

If we regard mg as a conservative force or depending solely upon position, as it really does, then F and N are conservative also. The initial potential energy with reference to p is then

$$P_1 = F \cos\beta \times d - mg \sin\alpha \times d.$$

The final potential energy is $P = 0$. We have then

$$P - P_1 = -F \cos\beta \times d + mg \sin\alpha \times d = 0, \quad \text{or} \quad F = \frac{\sin\alpha}{\cos\beta} mg.$$

In order to find N, suppose a virtual displacement d at right angles to F, so that the work of F during displacement is zero. Then the initial potential energy with reference to the point P is $P_1 = 0$. The final potential energy is

$$P = N \cos\beta \times d - mg \cos(\beta + \alpha) \times d.$$

We have then

$$P - P_1 = N \cos\beta \times d - mg \cos(\beta + \alpha) \times d = 0, \quad \text{or} \quad N = \frac{\cos(\beta + \alpha)}{\cos\beta} mg.$$

The same results are found by resolution of forces, page 173, Ex. 1, Vol. II, *Statics.*

If we regard mg as non-conservative, or not depending on position, as it practically is, then F and N are non-conservative also, and we have the principle of virtual work as given page 160, Vol. II, *Statics.* In this case there is no potential energy and the algebraic sum of the works of the non-conservative forces is zero.

Hence we have for the displacement $Pp = d$

$$-mg \sin\alpha \times d + F \cos\beta \times d = 0,$$

and for the displacement d perpendicular to F

$$+N \cos\beta \times d - mg \cos(\beta + \alpha) \times d = 0,$$

and evidently obtain the same values for F and N as before.

(2) *Take friction into account.*—Let the coefficient of static friction be μ. Then the friction is μN acting always in a direction opposite to the displacement along the plane.

Take mg as conservative or depending upon position Then N and F are conservative also. Suppose a virtual displacement d normal to the plane. Then the work of friction during displacement is zero. The initial potential energy is

$$P_1 = Nd + F \sin\beta \times d - mg \cos\alpha \times d.$$

The final potential energy is $P = 0$. We have then

$$P - P_1 = -Nd - F \sin\beta \times d + mg \cos\alpha \times d = 0,$$

or

$$N = mg \cos\alpha - F \sin\beta.$$

To find the force F necessary to just start the particle up the plane, suppose a displacement d up the plane. Then, since the friction μN is always opposite to the displacement, its work is $-\mu Nd$ and it is a non-conservative force. We have for the initial potential energy

$$P_1 = F \cos\beta \times d - mg \sin\alpha \times d.$$

The final potential energy is $P = 0$. Hence

$$P - P_1 = -F \cos\beta \times d + mg \sin\alpha \times d = -\mu Nd,$$

or

$$F \cos\beta = mg \sin\alpha + \mu N.$$

To find the force F necessary to just start the particle down the plane, suppose a displacement d down the plane. Then the friction μN is opposite to the displacement, and its work is $-\mu Nd$. We have for the initial potential energy

$$P_1 = +mg \sin\alpha \times d - F \cos\beta \times d,$$

and for the final potential energy $P = 0$. Hence

$$P - P_1 = -mg \sin\alpha \times d + F \cos\beta \times d = -\mu Nd,$$

or

$$F \cos\beta = mg \sin\alpha - \mu N.$$

These are the same results as found on page 215, Ex. 7, Vol. II, *Statics*, by resolution of forces.

If we regard mg as non-conservative or not depending on position, which it practically is, then F and N are non-conservative also. In this case there is no potential energy and the algebraic sum of the works of the non-conservative forces is zero.

Hence for a displacement d normal to the plane

$$+Nd + F \sin\beta \times d - mg \cos\alpha \times d = 0,$$

and we find N the same as before.

For a displacement d up the plane we have

$$F \cos\beta \times d - mg \sin\alpha \times d - \mu Nd = 0.$$

For a displacement d down the plane

$$-F\cos\beta \times d + mg\sin\alpha \times d - \mu Nd = 0.$$

These are evidently the same results as before.

EXAMPLES.

(1) *A ball weighing* 5 *ounces moving horizontally with a velocity of* 1000 *ft. per sec. strikes an obstacle, and after piercing it moves on with a velocity of* 400 *ft. per sec. Find the mean resistance of the obstacle if it is* 2 *inches thick.*

Ans. If the path is horizontal and at a distance h from the ground, the potential energy with reference to the ground is constant and equal to mgh. The initial energy is then $E_1 = \frac{1}{2}mv_1^2 + mgh$, and the final energy is $E = \frac{1}{2}mv^2 + mgh$. Hence

$$E - E_1 = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = Fs,$$

or, since $m = \frac{5}{16}$ lb., $v = 400$ ft. per sec., $v_1 = 1000$ ft. per sec., $s = \frac{2}{12}$ ft.,

$$\frac{1}{2}\cdot\frac{5}{15}\cdot 400^2 - \frac{1}{2}\cdot\frac{5}{16}\cdot 1000^2 = \frac{2}{12}F, \quad \text{or} \quad F = -787500 \text{ poundals,}$$

or F is equal to the weight of $-\frac{787500}{g}$ pounds. The minus sign indicates that F is opposite to the direction of motion.

If the path were vertically downwards we have initial potential energy mgh_1 and final potential energy mgh, and $h_1 - h = s$. Hence $E_1 = \frac{1}{2}mv_1^2 + mgh_1$, $E = \frac{1}{2}mv^2 + mgh$, $E - E_1 = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 - mg(h_1 - h) = Fs$, or

$$\frac{1}{2}\cdot\frac{5}{16}\cdot 400^2 - \frac{1}{2}\cdot\frac{5}{16}\cdot 1000^2 - \frac{5}{16}g\cdot\frac{2}{12} = \frac{2}{12}F, \text{ or } F = -\left(787500 + \frac{5}{16}g\right) \text{ pdls.,}$$

or F is equal to the weight of $-\left(\frac{787500}{g} + \frac{5}{16}\right)$ pounds. The minus sign indicates that F is opposite to the direction of motion.

Again, if the path were vertically upwards we have $h - h_1 = s$, and in the same way $F = -\left(787500 - \frac{5}{16}g\right)$ poundals, or equal to the weight of $-\left(\frac{787500}{g} - \frac{5}{15}\right)$ pounds.

(2) *A body of mass m is projected up an inclined plane whose inclination is α with a speed v_1. If the coefficient of kinetic friction is μ, find the space s in coming to rest.*

Ans. The normal pressure is $N = mg\cos\alpha$, and the friction is $\mu mg\cos\alpha$. Assume a horizontal plane through the starting-point. Then at the start the potential energy with reference to this plane is zero, and $E_1 = \frac{1}{2}mv_1^2$. At the end the kinetic energy is zero and the potential energy is $mgs\sin\alpha = E$. Hence

$$E - E_1 = mgs\sin\alpha - \frac{1}{2}mv_1^2 = -\mu mg\cos\alpha\,.\,s,$$

or

$$v_1^2 = 2gs(\sin \alpha + \mu \cos \alpha) \quad \text{and} \quad s = \frac{v_1^2}{2g(\sin \alpha + \mu \cos \alpha)}.$$

The general formula for uniformly accelerated motion is (page 51, Vol. I, *Kinematics*) $v^2 = v_1^2 + 2fs$. In the present case $v = 0$, and substituting the value of v_1^2 we have

$$2gs(\sin \alpha + \mu \cos \alpha) + 2fs = 0, \quad \text{or} \quad f = -g(\sin \alpha + \mu \cos \alpha).$$

The minus sign shows retardation, $-g \sin \alpha$ is the retardation due to the weight, and $-\mu g \cos \alpha$ is the retardation due to friction.

(3) *Find the height h to which a body weighing 2 lbs. and projected vertically upwards with a speed of 20 ft. per sec. will have risen before its speed is reduced to 5 ft. per sec., assuming the resistance of the air to be 10 lbs. per unit of distance described.*

Ans. Initial energy $E_1 = \frac{2 \times 20^2}{2}$. Final energy $E = \frac{2 \times 5^2}{2} + 2gh$.

$$E - E_1 = \frac{2 \times 5^2}{2} + 2gh - \frac{2 \times 20^2}{2} = -10gh, \quad \text{or} \quad h = \frac{375}{12g} \text{ ft.}$$

(4) *Find the speed v of a pendulum of length l which has swung from its extreme position through a given angle. Neglect all resistances and mass of the rod.*

Ans. Let θ be the angle made with the vertical in the extreme position, β be the angle in the position for which the speed is required. Take a horizontal plane through this position. At extreme position kinetic energy is zero and potential energy with reference to this plane is $mgl(\cos \beta - \cos \theta) = E_1$, where m is the mass of the bob. At final position potential energy is zero and kinetic energy is $\frac{1}{2}mv^2 = E$. Hence

$$E - E_1 = \frac{1}{2}mv^2 - mgl(\cos \beta - \cos \theta) = 0, \quad \text{or}$$

$$v = \sqrt{2gl(\cos \beta - \cos \theta)},$$

or the same as for a body falling freely through the distance $l(\cos \beta - \cos \theta)$.

(5) *A body of mass m slides down a smooth plane whose inclination is α. Show that the speed attained is the same as for falling through the vertical projection of the space described.*

Ans. Let s be the space described, then $s \sin \alpha$ is the vertical projection of this space. Take a horizontal plane at this distance below the starting-point. Then at start the kinetic energy is zero and the potential energy with reference to this plane is $mgs \sin \alpha = E_1$. At the end the potential energy is zero and the kinetic energy is $\frac{1}{2}mv^2 = E$. Hence

$$E - E_1 = \frac{1}{2}mv^2 - mgs \sin \alpha = 0, \quad \text{or} \quad v = \sqrt{2gs \sin \alpha}.$$

(6) *Compare the momentum and the kinetic energy in a mass of 20 lbs. having a speed of 16 ft. per sec., and a mass of 1 oz. having a speed of 5120 ft. per sec.*

Ans. Momentum is 320 *pound-velos* in both cases (page 32), or a constant resistance of 320 poundals will bring each mass to rest in one second. Kinetic energy in first case is 2560 ft.-poundals, and in the second case 819200 ft.-

poundals, or 320 times as much as in the first case. The constant resistance which will bring the first mass to rest in t seconds is then $\frac{320}{t}$ poundals, and the work whatever the time is 2560 ft.-poundals. The constant resistance which will bring the second mass to rest in t seconds is also $\frac{320}{t}$ poundals, and the work whatever the time is 819200 ft.-poundals.

(7) *A cannon-ball of* 5000 *grams is discharged with a speed of* 500 *metres per sec. Find the kinetic energy in ergs and in ft.-lbs.*

Ans. 6.25×10^{12} ergs; 4.61×10^{5} ft.-lbs., approximately.

(8) *A body weighing* 112 *lbs. is lifted* 20 *ft. Find the increase of potential energy.*

Ans. 2240 ft.-lbs.

(9) *A bow* 1 *yard long is straight when the string is just tight, but when bent has the form of a circular arc of* 1 *ft.* 6 *in. radius. The mean force exerted by the hand in bending per unit of distance through which it has moved is equal to the weight of* 10 *lbs. Find the potential energy of the bow.* ($g = 32$.)

Ans. 480 ft.-poundals.

(10) *A body is projected either vertically upwards or in any direction. Show, by calculating its kinetic and potential energy after any time, that in both cases the energy is the same at all points of its path. (Neglect the resistance of the air and assume g to have the same value at all points of the path.)*

(11) *A meteorite falls in a straight line towards the earth from a great distance. Show, by calculating the changes produced in its kinetic and potential energy between any two points of its path, that there is no change in its energy. (Neglect resistance of air.)*

(12) *A particle weighing* 1 *lb. has a simple harmonic motion with a period of* 20 *sec. and an amplitude of* 1 *ft. Find* (*a*) *its kinetic energy in its mean position,* (*b*) *its potential energy in either extreme position,* (*c*) *its kinetic and potential energy and their sum when at a distance of* 8 *inches from the mean position.*

Ans. (*a*) $\frac{\pi^2}{200}$ ft.-poundals; (*b*) the same; (*c*) kinetic energy $\frac{\pi^2}{360}$ ft.-poundals, potential energy $\frac{\pi^2}{450}$ ft.-poundals, their sum $\frac{\pi^2}{200}$ ft.-poundals.

(13) *What average force will bring to rest in* 100 *ft. a train of* 30 *tons* (2240 *lbs.*) *which has a speed of* 10 *miles an hour? Also what average force will bring it to rest in* 5 *seconds?*

Ans. 72277 poundals; 197120 poundals.

(14) *A horse-car of* 2240 *lbs. is stopped by a brake* 10 *times in going a mile; the brake stops the car in* 11 *yards; after each stoppage the car attains a speed of* 7½ *miles an hour. Supposing the friction to be a uniform force of* 28 *lbs., compare the work done by the horses with their work in going a mile with uniform speed of* 7½ *miles an hour, the track being level in both cases.*

Ans. The speed of 7½ miles an hour is 11 ft. per sec. No work is done by the horses while the brake is applied, that is, for 330 ft. The work done in producing kinetic energy is $\frac{2240}{2} \times 11^2 \times 10$ ft.-poundals. The work against friction is $28g \times 4950$ ft.-poundals.

Taking $g = 32$, the total work done is $2240 \times 11 \times 235$ ft.-poundals. The work in going a mile with uniform speed is $28 \times 32 \times 5280 = 2240 \times 11 \times 192$. These two are in the ratio of 235 to 192.

If the track rises 80 ft. in a mile, the work done in each case must be increased by $2240g \times 80$ ft.-poundals. If the track falls 80 ft. in a mile, the work done in each case must be diminished by the same amount.

(15) *Suppose the car in the preceding example has no brake, but must be stopped by the horses. How much more work would the horses have to perform?*

Ans. The friction is $28 \times 32 \times 330$ ft.-poundals. The work of destroying the kinetic energy is $\frac{2240}{2} \times 11^2 \times 10 = 2240 \times 55 \times 11$ ft.-poundals. The added work of the horses is

$$2240 \times 11(55 - 12) \text{ ft.-poundals}, \quad \text{or} \quad 70 \times 11(55 - 12) = 33110 \text{ ft.-lbs.}$$

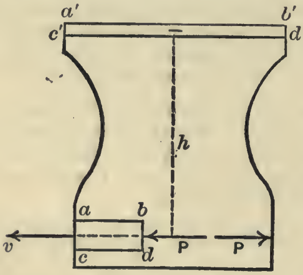
(16) *If the expense of moving a train is proportional to the work done, compare the cost of getting the speed of a train up from rest to* 45 *miles an hour and at the same time going a mile, with the cost of moving it a mile with that uniform speed. The resistance of friction being* 1/120 *the weight of the train. Track level.*

Ans. 163 to 64.

(17) *A vessel full of water has a small orifice at a distance h below the surface. Find the theoretic velocity of efflux.*

Ans. Let v be the velocity of efflux.

In a very small interval of time let a mass of water represented by $abcd$, whose centre of mass is in the horizontal through the centre of mass of the orifice, pass out, and let the surface sink from $a'b'$ to $c'd'$. The mass of water represented by $a'b'c'd'$ must be equal to the mass $abcd$. Call it m. If the orifice is very small compared to the area of cross-section of the vessel, the distance h between the centres of mass of $a'b'c'd'$ and $abcd$ will be practically equal to the distance from the top surface to the centre of mass of the cross-section of the orifice.



We have then potential energy of $a'b'c'd'$ equal to $mgh = E_1$, and kinetic energy of $abcd$ equal to $\frac{1}{2}mv^2 = E$. Hence, disregarding friction, $E - E_1 = 0$, or

$$mgh = \frac{1}{2}mv^2, \quad \text{or} \quad v = \sqrt{2gh}.$$

The theoretic velocity of efflux is the same as that obtained by a body falling freely through the distance from the top surface to the centre of mass of the orifice.

This is known as *Torricelli's principle.*

If a is the area of the orifice and v is the velocity, the quantity discharged in a very small interval of time τ is $av\tau$. If γ is the density or mass of a unit of volume, the mass discharged is $m = \gamma av\tau$. The kinetic energy of this mass is $\frac{1}{2}mv^2 = \frac{\gamma av\tau}{2}v^2$. The distance passed through by each particle from rest in attaining this velocity is $\frac{v}{2}\tau$. Hence, dividing the kinetic energy by the distance, we obtain the pressure $P = \gamma av^2$, or in gravitation units $P = 2\gamma a\frac{v^2}{2g}$.

But we have just seen that $\frac{v^2}{2g} = h$. Hence $P = 2\gamma ah$.

That is, *the pressure at the orifice is equal to twice the weight of a column of water whose base is the orifice and height h, or twice the static pressure when the orifice is closed.*

Since action and reaction are equal, this pressure P acts in the opposite direction upon the side of the vessel to move it.

(18) *In the preceding example let the same amount of water flow in at top as flows out.*

Ans. Let the water flow in at top with a velocity c, and let the top area be A. Then in any small interval of time the mass which enters is $\gamma Ac\tau$, and the kinetic energy at entrance is $\frac{\gamma Ac\tau}{2}c^2$. The potential energy at entrance is $\gamma Ac\tau \,.\, gh$. Hence $E_1 = \frac{\gamma Ac\tau}{2}c^2 + \gamma Ac\tau \,.\, gh$. The mass flowing out is $\gamma av\tau$, and its kinetic energy is $\frac{\gamma av\tau}{2}v^2 = E$. The two masses are equal, or

$$\gamma Ac\tau = \gamma av\tau, \quad \text{or} \quad Ac = av, \quad \text{or} \quad c = \frac{av}{A}.$$

We have then $E - E_1 = 0$, or

$$\frac{\gamma av\tau}{2}v^2 - \frac{\gamma Ac\tau}{2}c^2 = \gamma Ac\tau \,.\, gh, \quad \text{or} \quad h = \frac{v^2}{2g} - \frac{c^2}{2g}.$$

Inserting the value of $c = \frac{av}{A}$ and reducing, we have

$$v = \frac{\sqrt{2gh}}{\sqrt{1 - \frac{a^2}{A^2}}}.$$

We see that when a is very small compared to A, this reduces to $v = \sqrt{2gh}$ as in the preceding example. If the area of orifice $a = A$, v becomes infinity, that is, water must flow in and out with infinite velocity to make the orifice run full.

(19) *In the preceding example let the vessel move forward horizontally with a velocity* c_1, *while the water flows in at the top with a velocity* c, *and is discharged with the velocity* v, *making an angle* α *with the horizontal.*

In this case we have, just as in the preceding example,

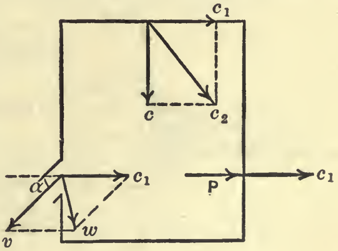
$$h = \frac{v^2}{2g} - \frac{c^2}{2g},$$

and if the water runs in as fast as it runs out

$$Ac = av.$$

The absolute velocity of the entering water is given by

$$c_2^2 = c_1^2 + c^2.$$



The absolute velocity of the departing water is given by

$$w^2 = c_1^2 + v^2 - 2c_1v\cos\alpha.$$

The mass of water entering and departing is $\gamma a v \tau = \gamma A c \tau$.

The kinetic energy of the water at entrance is $\frac{\gamma A c \tau}{2} \cdot c_2^2$. The kinetic energy at efflux is $\frac{\gamma a v \tau}{2} \cdot w^2$. The potential energy at entrance is $\gamma A c \tau h g$. Hence

$$E_1 = \frac{\gamma A c \tau}{2} c_2^2 + \gamma A c \tau h g \quad \text{and} \quad E = \frac{\gamma a v \tau}{2} \cdot w_2.$$

The energy transmitted to the vessel, or the work W of the resistance of the vessel, is $W = E - E_1$, or

$$W = \frac{\gamma a v \tau}{2} \cdot w^2 - \frac{\gamma A c \tau}{2} c_2^2 - \gamma A c \tau h g.$$

Substituting the values of c_2^2 and w^2, we have, since $\gamma A c \tau = \gamma a v \tau$, and $h = \frac{v^2}{2g} - \frac{c^2}{2g}$, for the work of the resistance of the vessel

$$W = - \gamma a v \tau \cdot c_1 v \cos \alpha.$$

The minus sign shows that the pressure of the vessel on the water is opposite to the motion.

The horizontal pressure in the direction of motion of the vessel is then found by dividing this by $c_1 \tau$, or

$$P = \gamma a v^2 \cos \alpha,$$

or, in gravitation units, $\frac{a v^2 \cos \alpha}{g}$.

If $\alpha = 0$, we have

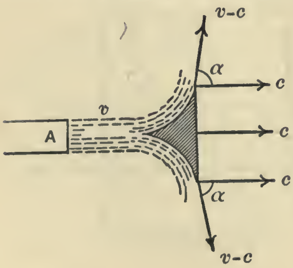
$$\frac{\gamma a v^2}{g} = 2\gamma a \frac{v^2}{2g} = 2\gamma a h.$$

This is the same result as in example (17).

The reaction of a horizontal jet is equal to the weight of a column of water whose cross-section is that of the stream and whose height is double that due to the velocity.

(20) *A stream of water whose cross-section is A and velocity v meets a surface moving in the same direction with the velocity c. Disregarding friction, what is the pressure in the direction of motion?*

Let the water pass off the surface in a direction making an angle α with the direction of motion of the surface.



The mass of water in any time τ is $\gamma A v \tau$, where A is the area of jet, v the velocity and γ the mass of a unit of volume.

The kinetic energy in the water before meeting the surface is then $E_1 = \frac{\gamma A v \tau}{2} \cdot v^2$.

The velocity of the water as it leaves the surface is $v - c$ relative to the surface. The velocity of the surface is c. The absolute velocity of the water as it leaves the surface is then given by the resultant of c and $v - c$, or

$$(v - c)^2 + c^2 + 2(v - c)c = w^2.$$

The kinetic energy of the departing water is $\frac{\gamma A v \tau}{2} \cdot w^2 = E$.

We have then $E - E_1 =$ work W of the resistance of the surface, or

$$W = -\frac{\gamma A v\tau}{2}(v^2 - w^2) = -\frac{\gamma A v\tau}{2}[2vc - 2c^2 - 2(v-c)c\cos\alpha]$$
$$= -\gamma A v\tau(1-\cos\alpha)(v-c)c,$$

and this is the work transmitted to the surface. The minus sign shows that the pressure of the surface on the water is opposite to the motion. Divide this by the distance $c\tau$, and we have for the pressure on the surface in the direction of motion

$$P = \gamma A v(1-\cos\alpha)(v-c),$$

or, in gravitation units,

$$P = \frac{\gamma A v}{g}(1-\cos\alpha)(v-c).$$

If the surface moves with a velocity c in the opposite direction, we have $v + c$ in place of $v - c$, and

$$P = \frac{\gamma A v}{g}(1-\cos\alpha)(v+c).$$

If the surface is at rest, $c = 0$, and

$$P = \frac{\gamma A v^2}{g}(1-\cos\alpha).$$

If in the latter case $\alpha = 90°$, this becomes

$$P = \frac{\gamma A v^2}{g} = 2\gamma A\frac{v^2}{2g}.$$

The normal pressure of water against a plane surface at rest is equal to the weight of a column of water whose cross-section is equal to the cross-section of the stream, and whose height is twice that due to the velocity of the stream.

If $\alpha = 180°$ and $c = 0$, we have

$$P = \frac{2\gamma A v^2}{g},$$

or twice as much as when α is 90°.

The work done on the surface is

$$\gamma A v\tau(1-\cos\alpha)(v-c)c.$$

This is a maximum when $c = v - c$ or $v = 2c$. That is, *the work done is a maximum when the velocity of the stream is twice that of the surface.* The maximum work is then

$$\gamma A v\tau(1-\cos\alpha)\frac{v^2}{4}.$$

If $\alpha = 180°$ this becomes $\frac{\gamma A v\tau}{2} . v^2$, or all the kinetic energy of the water.

If $\alpha = 90°$ it becomes $\frac{\gamma A v\tau}{4} . v^2$, or nearly one half the kinetic energy of the water.

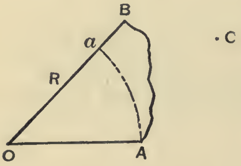
The maximum work of a stream of water striking a plane surface at right angles, disregarding friction, is only one half the kinetic energy of the water.

CHAPTER VIII.

THE POTENTIAL.*

PRINCIPLE OF THE POTENTIAL. EQUIPOTENTIAL SURFACE. LINES AND TUBES OF FORCE. GRAVITATIONAL POTENTIAL. DIFFERENTIAL EQUATIONS. THEOREM OF LA PLACE. POISSON'S EXTENSION.

The Potential.—Let a particle at a fixed point O act either by attraction or repulsion upon a particle at B. Let BA be any path of the particle from B to A, the distance OB being R and the distance OA being r. With $OA = r$ as a radius describe an arc of a circle Aa.



Then the force upon B is a central force, and we have proved, page 46, that the work done by or against the central force while the particle moves from B to A is *independent of the path* and equal to that necessary to move it from B to a, when $Oa = r$.

The fixed particle at O is then a centre of force, and the space surrounding this particle we call the **field of force.**

If then we take any convenient point of reference as C, the work done in transferring a particle of unit mass from any point of the field to this point, or from this point to any point of the field, has a definite value for every point of the field, no matter what the path.

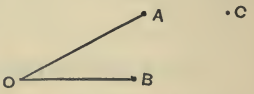
This definite work for any given point of the field *when the particle moved has a mass of unity* is called the **potential** of the point.

The unit of potential is then the same as the unit of work, as one foot-poundal or one foot-pound or one erg.

The magnitude of the potential will depend upon the position of the point of reference. The sign will be plus or minus, according as work is done by or against the force of the field. The potential is usually denoted by the letter V.

Principle of the Potential.—The application of the potential rests upon the following principle.

Let A and B be any two points in the field of force due to a particle at O, and let C be any point of reference. Then since the work done during any displacement is independent of the path, the work done by or against the force of the field in transferring a unit mass from A to B is equal to the difference of the works done in transferring it from A to C and C to B.



If then V_A and V_B are the potentials of the points A and B, the difference $V_A \sim V_B$ is the work of moving unit mass from A to B or B to A. If F is the mean force in the direction AB, we have this work equal to $F \times AB$. Hence

$$F \times AB = V_A \sim V_B, \quad \text{or} \quad F = \frac{V_A \sim V_B}{AB}.$$

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* This chapter must be omitted by those not familiar with the calculus.

When A and B are indefinitely near, the mean force F becomes the instantaneous force in the direction AB, and $\frac{V_A \sim V_B}{AB}$ becomes the rate of change of the potential of the point A per unit of distance in the direction AB. Hence,

The rate of change of the potential of any point per unit of distance in any direction is equal to the component force in that direction which acts upon a particle of unit mass placed at that point.

The particle possesses potential energy at whatever point of the field of force it may be placed. The excess of its potential energy at one point over its potential energy at another point is then the work done by or against the force of the field in moving from one point to the other. This is equal to the difference of potential. Hence the appropriateness of the term "potential."

The theory of the potential is of great use in magnetic and electrical investigations.

Equipotential Surface.—A surface at every point of which the potential has the same value is called an **equipotential surface.**

If then a particle is moved from any point on such a surface to any other point on this surface no work is done by or against the force of the field. There is then no component force in any direction tangential to such a surface, and hence no rate of change of potential per unit of distance in that direction. The resultant force at any point of such a surface is then normal to the surface. Thus the surface of water at rest forms an equipotential surface for which there is no rate of change of potential, and the resultant force for every particle on the surface is normal to the surface. The work done by or against gravity in moving a particle from one point to another of such a surface is zero.

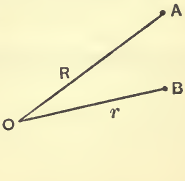
Lines of Force.—Any line so drawn in a field of force that its direction at every point is the direction of the resultant force at that point is called a **line of force.** As the resultant force at any point is normal to the equipotential surface passing through that point, lines of force are normal to the equipotential surfaces they meet.

Tubes of Force.—If from points in the boundary of any portion of an equipotential surface lines of force are drawn, the space thus marked off is called a **tube of force.**

Gravitational Potential.—The choice of the point of reference and of the mode of defining potential are matters of convenience and vary with the kind of field of force under consideration.

The potential in a field of force due to the attraction of gravity is called the **gravitational potential.** The point of reference is taken in this case *at an infinite distance*, and since it is convenient to have the potential for all points of a gravitational field positive, and the force of the field is always attractive, we define gravitational potential of a point as *the work done* BY *the force of the field in moving unit mass from a point at an infinite distance* TO *the given point.* Or, since there is thus a loss of potential energy, the work done by the force of the field must equal the gain of kinetic energy, and hence we may also define gravitational potential of a point as *the kinetic energy acquired by unit mass in falling from infinity under the attraction of a given mass to that point.*

The force of gravity varies inversely as the square of the distance, and we have seen (page 47) that the work of such a force when a particle moves from a distance R to a distance r from the centre of force is given by



$$W = F'r'^2\left(\frac{1}{r} - \frac{1}{R}\right),$$

where F' is the force at a given distance r'.

If we take R infinite, $\frac{1}{R}$ is zero and this becomes

$$W = F'r'^2 . \frac{1}{r}.$$

If the mass of the attracting particle at O is m and of the moving particle M, we have for the force of attraction at any distance r' (page 44, Vol. II, *Statics*)

$$F' = \kappa \frac{Mm}{r'^2},$$

where κ (page 48, Vol. II, *Statics*) is given by

$$\kappa = \frac{gr'^2}{m'},$$

where g is the acceleration of gravity, m' the mass of the earth and r' the radius of the earth. We have then for the work of moving a particle M from infinity to the distance r

$$W = \kappa M \frac{m}{r}.$$

If we adopt the astronomical unit of mass (page 48, Vol. II, *Statics*) this becomes $W = M\frac{m}{r}$, and if we take the mass M as unity we have $W = [M]\frac{m}{r}$, where $[M]$ is the unit of mass, or the numerical equation $W = \frac{m}{r}$.

If the field of force is due to any number of particles of masses m_1, m_2, m_3, etc., at distances r_1, r_2, r_3, etc., from M, we have the numeric equation when M is unity

$$W = \Sigma \frac{m}{r}.$$

The expression $\Sigma \frac{m}{r}$ is the gravitational potential of the point at which the attracted particle of unit mass is placed. We have then

$$V = \Sigma \frac{m}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, mathematically defined, *the gravitational potential of any point due to the attraction of any mass is the sum of the quotients of all the elementary attracting masses divided by their distances from the point.*

Since equation (1) gives the work done by the force of the field in moving *unit mass* from a point at an infinite distance to the given point, the work done in moving a mass M is

$$W = M\Sigma \frac{m}{r} = MV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if we use the astronomical unit of mass (page 48, Vol. II, *Statics*), or

$$W = M\Sigma \frac{m}{r} = \kappa MV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

if we use the ordinary unit of mass, where κ is given by

$$\kappa = \frac{gr'^2}{m'}, \quad \ldots \ldots \ldots \ldots \quad (4)$$

where g is the acceleration of gravity, m' the mass and r' the radius of the earth (page 48, Vol. II, *Statics*).

Differential Equations.—We have then for the gravitational potential of any point of a field of force due to the attraction of any number of particles m_1, m_2, m_3, etc., at distances r_1, r_2, r_3 from that point,

$$V = \Sigma\frac{m}{r}. \quad \ldots \ldots \ldots \ldots \quad (1)$$

From the principle of the potential (page 102), if we take the point as an origin of co-ordinates, we have for the component force in the directions of the axes of X, Y, Z, for a *unit mass* at the point,

$$\left.\begin{aligned} F_x &= \frac{dV}{dx}; \\ F_y &= \frac{dV}{dy}; \\ F_z &= \frac{dV}{dz}; \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (2)$$

where the astronomical unit of mass (page 48, Vol. II, *Statics*) is to be used. For the ordinary unit of mass we multiply by

$$\kappa = \frac{gr'^2}{m'}, \quad \ldots \ldots \ldots \ldots \quad (3)$$

where g is the acceleration of gravity, m' the mass and r' the radius of the earth (page 48, Vol. II, *Statics*).

For a mass M, then, at the point we multiply by κM.

For the resultant force on *unit of mass* in the direction of any radius vector from the point we have

$$R = \frac{dV}{dr}, \quad \ldots \ldots \ldots \ldots \quad (4)$$

where the astronomical unit of mass (page 48, Vol. II, *Statics*) is to be used. For ordinary unit of mass we multiply by κ, and for any mass M at the point by κM.

If ds is an element of the path of the attracted particle of unit mass, making an angle θ with r, then $ds = \frac{dr}{\cos\theta}$, and we have for the component of the force tangent to the path, upon *unit mass*,

$$F_t = \frac{dV}{ds} = \frac{dV}{dr}\cos\theta, \quad \ldots \ldots \ldots \ldots \quad (5)$$

where the astronomical unit of mass (page 48, Vol. II, *Statics*) is to be used.

For ordinary unit of mass we multiply by κ, and for any mass M by κM.

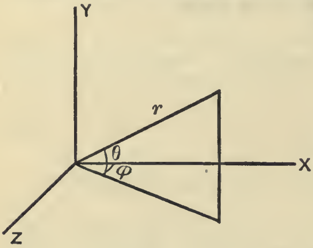
We have from (4), $V = \int R dr$; and since for an equipotential surface the potential has the same value at every point, the condition for an equipotential surface is

$$V = \int R dr = C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where C is a constant.

A surface which fulfils for each of its points this condition is an equipotential surface for the system of attractions. As any value can be given to C between its greatest and least values, there will be an indefinitely great number of equipotential surfaces corresponding to any given system of attractions.

From equation (5) we see that F_t is zero when $\theta = 90°$, and becomes equal to the resultant force R, equation (4), when $\theta = 0$. That is, the resultant attraction R is at right angles to the equipotential surface. The direction of R is then a line of force.



If dv is an element of volume, and δ its density, we have for its mass $m = \delta dv$. Hence for rectangular co-ordinates

$$V = \int \frac{m}{r} = \int \frac{\delta dxdydz}{r} . \quad . \quad . \quad (7)$$

If we use polar co-ordinates we have for the elementary volume (page 40, Vol. II, *Statics*) $dv = r^2 dr \cos\theta \, d\theta \, d\phi$, and hence

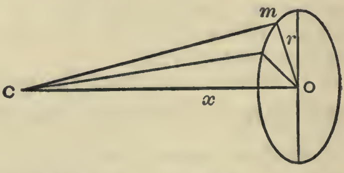
$$V = \int\int\int \delta r \, dr \cos\theta \, d\theta \, d\phi. \quad (8)$$

EXAMPLES.

(1) *Particles of masses* 3.928, 39.28 *and* 392.8 *kilograms are situated at three of the corners of a square whose side is* 1 *metre. Find the potential at the fourth corner.*

Ans. $V = \Sigma \frac{m}{r}$, and the astronomical unit of mass is 3928 grams (page 48, Vol. II, *Statics*). Hence $V = 1.087$ ergs.

(2) *Find the potential and attraction of a homogeneous circular ring of radius r upon a point C on the perpendicular to its plane through its centre O.*



Ans. Let the distance of the point C from the centre O be x. Then the distance Cm for any particle of the ring is $\sqrt{r^2 + x^2}$. If the linear density of the ring is δ, the mass is $2\pi r\delta$, and therefore the potential $V = \frac{2\pi r\delta}{\sqrt{r^2 + x^2}}$.

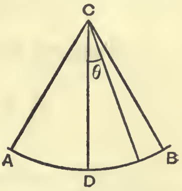
The attraction upon a unit mass at C parallel to the plane of the ring is then $\frac{dV}{dr}$, taking the astronomical unit of mass (page 48, Vol. II, *Statics*). But r is constant, and hence $\frac{dV}{dr} = 0$. That is, the sum of the component attractions of the elements of the ring in the plane of the ring is zero. The attraction in the direction CO upon a unit mass at C, taking the astronomical unit of mass, is $A_x = \frac{dV}{dx} = -\frac{2\pi r\delta x}{(r^2 + x^2)^{\frac{3}{2}}}$, the minus sign denoting attraction or force towards the centre O. This is the same result as already found, page 51, Vol. II, *Statics*.

If we multiply the value of V and A_x by κM, where M is the mass of any particle at O, and κ is $\frac{gr'^2}{m'}$ (page 48, Vol. II, *Statics*), we have the result for any mass M at C, using the ordinary unit of mass.

When $x = 0$ the potential at the centre of the ring is $V = 2\pi\delta$.

(3) *Find the potential and attraction of a circular arc at its centre.*

Ans. Let θ be the angle subtended by any portion of the arc estimated from its middle point D.



The length of any element is $rd\theta$, its mass is $r\delta d\theta$, where δ is the linear density, and the potential is

$$V = \int_{-\alpha}^{+\alpha} \frac{r\delta d\theta}{r} = 2\delta\alpha,$$

where α is the angle ACD.

This is independent of the radius r of the arc.

The attraction of any element whose mass is $r\delta d\theta$ for a unit mass at C, using the astronomical unit of mass (page 48, Vol. II, *Statics*), is $\frac{r\delta d\theta}{r^2}$. The component of this at right angles to CD is $\frac{r\delta d\theta}{r^2}\sin\theta$, and along CD, $-\frac{r\delta d\theta}{r^2}\cos\theta$.

We have then for the resultant attraction at right angles to CD

$$A_x = \frac{\delta}{r}\int_{-\alpha}^{+\alpha} d\theta \sin\theta = 0,$$

and for the resultant attraction along CD

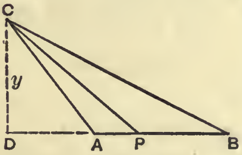
$$A_y = -\frac{\delta}{r}\int_{-\alpha}^{+\alpha} d\theta\cos\theta = -\frac{2\delta}{r}\sin\alpha,$$

the minus sign denoting attraction.

This is the same result as already found, page 50, Vol. II, *Statics*, for a unit mass at C, using the astronomical unit of mass. For any mass M at C, using the ordinary unit of mass, we multiply by κM, where $\kappa = \frac{gr'^2}{m'}$ (page 48, Vol. II, *Statics*).

(4) *Find the potential and attraction of a straight line upon an external point.*

Ans. Let AB be the line and C the point. Drop the perpendicular CD, take D as origin, and let $CD = y$. Then for any point P of the line distant $DP = x$ we have $CP = r = \sqrt{y^2 + x^2}$. Let δ be the linear density. Then the mass of any element is δdx, and the potential is



$$V = \delta\int\frac{dx}{\sqrt{y^2+x^2}} = \delta\log\frac{x+\sqrt{y^2+x^2}}{y}.$$

Taking this between the limits of $x = DA = +a$ and $x = DB = +b$, we have

$$V = \delta \log \frac{b + \sqrt{y^2+b^2}}{a + \sqrt{y^2+a^2}}.$$

The component attraction upon unit mass at C in the direction of the line is

$$F_x = \frac{dV}{dx} = \frac{\delta}{\sqrt{y^2+x^2}}.$$

Introducing the limits $+a$ and $+b$,

$$F_x = \delta\left(\frac{1}{\sqrt{y^2+b^2}} - \frac{1}{\sqrt{y^2+a^2}}\right) = \delta\left(\frac{1}{CB} - \frac{1}{CA}\right).$$

For the component attraction upon the unit mass at C perpendicular to the line we have

$$F_y = \frac{dV}{dy} = \delta \cdot \frac{d\left[\log\left(\frac{x}{y} + \sqrt{1+\frac{x^2}{y^2}}\right)\right]}{dy}$$

$$= -\delta \cdot \frac{\frac{x}{y^2} + \frac{yx^2}{y^4\sqrt{1+\frac{x^2}{y^2}}}}{\frac{x}{y} + \sqrt{1+\frac{x^2}{y^2}}} = -\frac{\delta x}{y\sqrt{y^2+x^2}}.$$

Introducing the limits $+a$ and $+b$, we have

$$F_y = \frac{\delta}{y}\left(\frac{a}{\sqrt{y^2+a^2}} - \frac{b}{\sqrt{y^2+b^2}}\right) = \frac{\delta}{y}\left(\frac{a}{CA} - \frac{b}{CB}\right).$$

Let the angle $DCA = \alpha$, $DCB = \beta$, $ACB = \beta - \alpha = \gamma$. Then

$$\frac{1}{CB} = \frac{\cos\beta}{CD}, \quad \frac{1}{CA} = \frac{\cos\alpha}{CD}, \quad a = CA \sin\alpha, \quad b = CB \sin\beta,$$

and

$$F_x = \frac{\delta}{CD}\left(\cos\beta - \cos\alpha\right), \quad F_y = \frac{\delta}{CD}\left(\sin\alpha - \sin\beta\right).$$

The resultant force upon unit mass at C is then

$$R = \sqrt{F_x^2 + F_y^2} = \frac{\delta}{y}\sqrt{2 - 2\cos\gamma} = \frac{2\delta}{y}\sin\frac{1}{2}\gamma.$$

This is the same result as already obtained, page 51, Vol. II, *Statics*.

The tangent of the angle which this resultant force makes with the vertical is

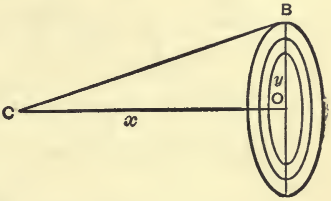
$$\frac{F_x}{F_y} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \tan\frac{\alpha+\beta}{2}.$$

Therefore the resultant attraction bisects the angle ACB.

The results are all for unit mass at C and astronomical unit of mass (page 48, Vol. II, *Statics*). For mass M at C and ordinary unit of mass we have only to multiply F_x, F_y, R by κM, where $\kappa = \frac{gr'^2}{m'}$ (page 48, Vol. II, *Statics*).

(5) *Find the potential and attraction for a circular disk at a point on the perpendicular to its plane through its centre.*

Ans. Let y be the radius and dy the thickness of an elementary ring, and δ the surface density. Then the mass of the elementary ring is $2\pi\delta y\,dy$ If the distance OC is x, we have for the potential of the disk



$$V = 2\pi\delta \int \frac{y\,dy}{\sqrt{x^2+y^2}}$$

$$= 2\pi\delta(\sqrt{x^2+y^2} - x),$$

which for the limits $y = R$ = radius of disk, and $y = 0$ becomes

$$V = 2\pi\delta(\sqrt{x^2 + R^2} - x).$$

For the centre of the disk this becomes $2\pi\delta R$.

The potential then is constant for x constant. The component force upon unit mass at C parallel to the disk is then $\frac{dV}{dR} = 0$. For the component force along OC we have $F_x = \frac{dV}{dx} = -2\pi\delta\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)$, the minus sign denoting attraction.

This is the same result as already obtained, page 52, Vol. II, *Statics*.

For mass M at O and ordinary unit of mass we have only to multiply F_x by κM, where $\kappa = \frac{gr'^2}{m}$ (page 48, Vol. II, *Statics*).

(6) *Find the potential and attraction at the vertex for a right cone with circular base.*

Ans. Let the half angle at the vertex, OCB, of the preceding figure be θ. Then $\frac{x}{\sqrt{x^2 + R^2}} = \cos\theta$. Hence from the preceding example we see that the attraction of all circular elementary slices for a particle at C is the same, and equal to

$$-2\pi\delta dx(1 - \cos\theta).$$

The total attraction is then

$$F_x = \frac{dV}{dx} = -2\pi\delta x(1 - \cos\theta),$$

which for the limits h and 0 becomes

$$F_x = -2\pi\delta h(1 - \cos\theta).$$

This is the same result as already obtained, page 52, Vol. II, *Statics*.

For mass M at C and ordinary units of mass we have only to multiply by κM, where $\kappa = \frac{gr'^2}{m'}$ (page 48, Vol. II, *Statics*).

We have then

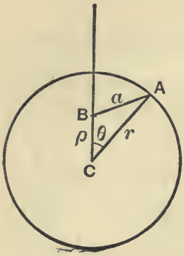
$$V = -\pi\delta x^2(1 - \cos\theta),$$

or for limits 0 and h

$$V = \pi\delta h^2(1 - \cos\theta).$$

(7) *Find the potential and attraction of a spherical shell at any point.*

Ans. Let r be the radius of the shell, t its thickness, ρ the distance of the point B from the centre C, $AB = a =$ the distance of any point of the shell from the given point B. Take the origin at C, and let BC coincide with the axis of Y.



Then (page 40, Vol. II, *Statics*) the elementary volume is

$$dv = r^2 t \sin\theta d\theta d\phi,$$

and

$$a = \sqrt{r^2 + \rho^2 - 2r\rho\cos\theta}.$$

Hence, if δ is the density,

$$V = \delta t r^2 \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta\, d\theta\, d\phi}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}}.$$

Integrating first with respect to ϕ, we have

$$V = 2\pi\delta t r^2 \int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}},$$

and then with respect to θ,

$$V = \frac{2\pi\delta t r}{\rho}\left\{(r^2 - 2r\rho\cos\theta + \rho^2)^{\frac{1}{2}}\right\}_0^{\pi}$$

$$= \frac{2\pi\delta t r}{\rho}\left[(r^2 + 2r\rho + \rho^2)^{\frac{1}{2}} - (r^2 - 2r\rho + \rho^2)^{\frac{1}{2}}\right].$$

When the point B is within the shell $\rho < r$, and when it is outside of the shell $\rho > r$.

In the first case, when B is within the shell, we have

$$V = \frac{2\pi\delta t r}{\rho}\left[(r + \rho) - (r - \rho)\right] = 4\pi\delta t r = \frac{m}{r},$$

where m is the mass of the shell. The resultant force of attraction is then $R = \frac{dV}{d\rho} = 0$. This is the same result as in example (3), page 54, Vol. II, *Statics.*

In the second case, when B is outside the shell, we have

$$V = \frac{2\pi\delta t r}{\rho}\left[(r + \rho) - (\rho - r)\right] = \frac{4\pi\delta t r^2}{\rho} = \frac{m}{\rho},$$

where m is the mass of the shell. The resultant force of attraction then is $R = \frac{dV}{d\rho} = -\frac{m}{\rho^2}$, where the minus sign denotes attraction.

If we take the mass M at B and use the ordinary unit of mass, we have $R = -\kappa\frac{Mm}{\rho^2}$. This is the same result as already obtained page 46, Vol. II, *Statics.*

(8) *Find the potential and attraction of a thick homogeneous spherical shell at any point.*

Ans. Let the external radius be r_1 and the internal radius r_2. Then in the preceding example we can put $t = dr$, and we have for the potential of that part of the shell outside of the spherical surface containing the point $\int_\rho^{r_1} 4\pi\delta r\,dr$, and for the potential of that part of the shell inside of the spherical surface containing the point $\int_{r_2}^{\rho} \frac{4\pi\delta r^2\,dr}{\rho}$. Hence

$$V = \frac{4\pi\delta}{\rho}\int_{r_2}^{\rho} r^2 dr + 4\pi\delta\int_{\rho}^{r_1} r dr$$

$$= \frac{4\pi\delta}{3\rho}(\rho^3 - r_2^3) + 2\pi\delta(r_1^2 - \rho^2).$$

The mass of the shell is $m = \frac{4\pi\delta}{3}(r_1^3 - r_2^3)$.

If the point is wholly within the shell,

$$V = 2\pi\delta(r_1^2 - r_2^2);$$

and if the thickness is very small, $r_1 - r_2 = t$ and $r_1 + r_2 = 2r$, and $V = 4\pi\delta tr$, as found in the preceding example. Also the attraction is $R = \frac{dV}{d\rho} = 0$, as found in the preceding example.

If the point is wholly without the shell,

$$V = \frac{4\pi\delta}{3\rho}(r_1^3 - r_2^3) = \frac{m}{\rho},$$

and $R = -\frac{m}{\rho^2}$, as found in the preceding example.

If the shell becomes a sphere, $r_2 = 0$ and $r_1 = r$, and we have for an interior point

$$V = 2\pi\delta r^2 - \frac{2\pi\delta\rho^2}{3};$$

$$R = -\frac{4\pi\delta\rho}{3} = -\frac{m}{r^3}\rho,$$

where $m = \frac{4}{3}\pi r^3\delta$. For an exterior point

$$V = \frac{4\pi\delta r^3}{3\rho} = \frac{m}{\rho};$$

$$R = -\frac{m}{\rho^2}.$$

For $\rho = r$ we have in both cases

$$V = \frac{4\pi\delta r^2}{3} = \frac{m}{r}; \quad R = -\frac{m}{r^2}.$$

Hence we see that for a homogeneous sphere we may take the potential and attraction at any external point as though the whole mass were concentrated at the centre; while the attraction at an interior point is directly proportional to the distance from the centre. The first result has been proved, page 46, Vol. II, *Statics;* the second in example (4), page 54, Vol. II, *Statics*.

(9) *Find the potential and attraction for a cylinder of length l and radius R for a point on the axis at a distance d from the nearest end.*

Ans. We have found, example (5), for the component force along the axis of a circular disk $-2\pi\delta\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$; the component at right angles to the axis being zero. If the disk has a thickness dx, we have for a cylinder

$$\frac{dV}{dx}=F_x=-2\pi\delta\int dx\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)=-2\pi\delta\left(x-\sqrt{x^2+R^2}\right);$$

or, taking the limits $d+l$ and d,

$$F_x=-2\pi\delta\left(l-\sqrt{(d+l)^2+R^2}+\sqrt{d^2+R^2}\right).$$

This is the same result as found on page 52, Vol. II, *Statics*. Hence

$$V=\int-2\pi\delta dx\left(x-\sqrt{x^2+R^2}\right)$$

$$=-2\pi\delta\left[\frac{x^2}{2}-\frac{x}{2}\sqrt{x^2+R^2}-\frac{R^2}{2}\log\left(x+\sqrt{x^2+R^2}\right)\right].$$

The value of V is obtained by taking the limits $d+l$ and d. For $d=0$ we have for the attraction upon unit mass at the end surface

$$F_x=-2\pi\delta\left(l-\sqrt{l^2+R^2}+R\right),$$

and

$$V=2\pi\delta\left[\frac{l}{2}\sqrt{l^2+R^2}-\frac{l^2}{2}-\frac{R^2}{2}\log\left(l+\sqrt{l^2+r^2}\right)\right].$$

The value for F_x is the same as found on page 52, Vol. II, *Statics*.

For a mass M, using the ordinary unit of mass, we multiply F_x by κM, where $\kappa=\frac{gr'^2}{m'}$ (page 48, Vol. II, *Statics*).

(10) *If the radius of the earth is* 4000 *miles, find the potential for a point on the surface.*

Ans. From example (8) we have for astronomical unit of mass $V=\frac{m'}{r'}$. For ordinary unit of mass we multiply by $\frac{gr'^2}{m'}$. Hence $V=gr'$ ft.-poundals, or gr ft.-lbs. $=4000\times5280=21120000$ ft.-lbs.

(11) *Show that the dimensions of potential are* $\frac{[M][L]^2}{[T]^2}$.

(12) *In a series of concentric spherical equipotential surfaces show that the distance between any two is proportional to the square of the geometric mean of the distances from the centre.*

Ans. Let r_1 and r_2 be the distances from the centre, r_1 the greater. Then if the field of force is due to the attraction of a particle of mass m at the centre, the potential for any point on the first surface is $V_1 = \frac{m}{r_1}$, and for any point on the second surface $V_2 = \frac{m}{r_2}$. The work done upon unit mass in passing from one surface to the other is $W = V_2 - V_1 = \kappa m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \kappa m \cdot \frac{r_1 - r_2}{r_1 r_2}$, where $\kappa = \frac{gr'^2}{m'}$ (page 48, Vol. II, *Statics*). Therefore $r_1 - r_2 = \frac{W}{\kappa m} \cdot r_1 r_2$, or the distance $r_1 - r_2$ between the surfaces is proportional to $r_1 r_2$. But if x is a geometric mean between r_1 and r_2 we have $r_1 : x :: x : r_2$, or $x^2 = r_1 r_2$, or $x = \sqrt{r_1 r_2}$. Therefore $r_1 r_2$ is the square of the geometric mean of the distances from the centre.

It follows that at great distances from the centre of the earth the unit mass must be moved a long distance in order to do a ft.-lb. of work.

(13) *A point A near the earth's surface is h feet above another such point at B. Find the excess of the potential of A over that of B.*

Ans. From preceding example $V_A - V_B = \kappa m' \frac{r_2 - r_1}{r_1 r_2}$, where $\kappa = \frac{gr'^2}{m'}$. We have $r_1 - r_2 = h$, and if the points are near the earth's surface $r_1 r_2$ approximately equal to r'^2. Hence $V_A - V_B = -gh$.

(14) *At the distance of the moon, 240000 miles from the earth's centre, find the shortest distance through which 1 lb. must be moved to do 1 ft.-lb. of work.*

Ans. From example (12) we have $W = \kappa m' \frac{r_1 - r_2}{r_1 r_2}$, or, inserting the value of κ, $W = gr'^2 \frac{r_1 - r_2}{r_1 r_2}$, where W is the work in ft.-poundals. For the work in ft.-lbs. we have $W = r'^2 \frac{r_1 - r_2}{r_1 r_2}$, or if $W = 1$ ft.-lb., $r_1 - r_2 = \frac{r_1}{1 + \frac{r'^2}{r_1}}$.

Taking $r' = 4000$ miles and $r_1 = 240000$ miles, we have

$$r_1 - r_2 = \frac{240000 \times 5280}{1 + \frac{4000 \times 4000 \times 5280 \times 5280}{240000 \times 5280}} = 3600 \text{ ft.}$$

The Theorem of La Place.—If any closed surface in a field of force is divided into small portions, the sum of the products of the areas of these portions by the normal components of the forces exerted at them on unit mass is called the integral normal attraction over the surface.

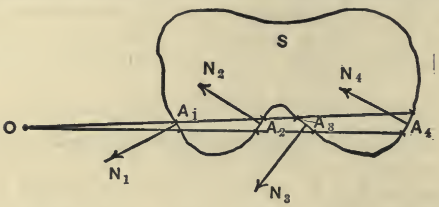
Thus if V is the potential for any point of the surface, then if ρ is the normal to the surface the normal force or unit mass is $N = \frac{dV}{d\rho}$. If dA is the small elementary area, the product NdA is the normal attraction over the elementary surface and $\int NdA = \int \frac{dV}{d\rho} dA$ is the integral normal attraction over the whole surface.

Let A be the area of any closed surface, and m any attracting mass wholly external to A. Then it can be shown that

$$\int NdA = \int \frac{dV}{d\rho} dA = 0.$$

This is known as the theorem of *La Place.*

Take a particle of mass m at O wholly outside the closed surface S. Consider the elementary cone which has dA for its base, and vertex at O.



If r be the length of the cone and $\delta\omega$ its solid angle, its base is $dA = r^2 d\omega \sec\phi$, where ϕ is the angle which the base makes with a normal section of the cone. The attraction is $\frac{m}{r^2}$ on unit mass, and the normal attraction $\frac{m}{r^2}\cos\phi = N$. Hence $NdA = md\omega$.

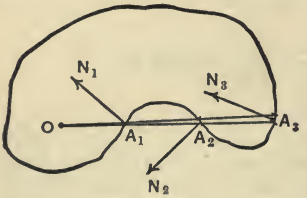
If the mass m at O is wholly *without* the surface, every line drawn from it will meet the surface in an even number of points. If we take the normal attraction as positive if directed outwards and negative if directed inwards, we have

$$+ N_1 dA_1 = - N_2 dA_2 = + N_3 dA_3 = - N_4 dA_4 = md\omega.$$

Hence

$$\int NdA = \int \frac{dV}{d\rho} dA = 0.$$

Poisson's Extension of La Place's Theorem.—If the point O is *inside the surface*, the cone whose vertex is O will cut the surface in whatever direction it is drawn an odd number of times. We have then



$$- N_1 dA_1 = + N_2 dA_2 = - N_3 dA_3 = md\omega.$$

Hence

$$\int NdA = \int \frac{dV}{d\rho} dA = \int md\omega = - 4\pi M',$$

where M' is the entire mass within the surface.

This is known as Poisson's extension of La Place's theorem.

Hence, combining the two theorems, we see that the sum of the attractions of any mass M estimated along the normals at all points of a closed surface is zero when the attracting matter M is wholly external to the surface and equal to $- 4\pi M'$ when the closed surface contains any portion M' of M, or

$$\int NdA = \int \frac{dV}{d\rho} dA = 0, \quad \text{or} \quad - 4\pi M'.$$

Let x, y, z be the co-ordinates of the attracting particle of density δ. Then we can also write

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0, \quad \text{or} \quad 4\pi\delta.$$

Thus, let the surface be a rectangular parallelopipedon whose sides are dx, dy, dz. Let N_x, N_y, N_z be the normal attractions on the sides, taken as positive in the positive direction of x, y, z.

For the two faces which are perpendicular to the axis of X we have $dA = dy\,dz$, and for a point within the surface we have for the face on the left $N_x = \frac{dV}{dx}$, and for the face on the right $-N_x = \frac{dV}{dx} + \frac{d^2V}{dx}$. Hence $\int N_x dA$ over this pair of faces is $-\frac{d^2V}{dx^2}dx\,dy\,dz$. For the other pairs of faces we have in the same way $-\frac{d^2V}{dy^2}dx\,dy\,dz$ and $-\frac{d^2V}{dz^2}dx\,dy\,dz$. Equating then the value of $\int NdA$ for the whole surface to $-4\pi M'$, that is, to $-4\pi dx\,dy\,dz\delta$, we have

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 4\pi\delta.$$

If the point is external, $\delta = 0$ and

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0.$$

EXAMPLES.

(1) *To find the attraction of a sphere or spherical shell of matter symmetrically distributed round the centre.*

Ans. By symmetry the attraction is the same at all points of a spherical surface and is in the direction of the radius drawn inwards. Let r be the radius of such a surface. Then

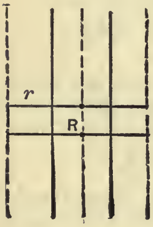
$$\int NdA = N\int dA = N.4\pi r^2.$$

Equating this to $-4\pi M'$, where M' is the mass within the surface, we have for the attraction on unit mass at distance r, using the astronomic unit of mass (page 48, Vol. II, *Statics*)

$$N = -\frac{M'}{r^2}.$$

N in this case is the whole attraction on unit mass at a distance r from the centre, and the result shows that it is the same as the attraction of a mass M' collected at the centre. If r is greater than or equal to the radius of the sphere or the external radius of the shell, M' will be the whole mass of the sphere or shell. If r is equal to or less than the internal radius of the shell, M' is zero. (See examples (7) and (8), pages 110, 111.)

(2) *To find the attraction of a cylinder, either solid or hollow, of indefinitely great length, the density being a function of the distance from the axis.*



Ans. By symmetry the attraction is the same at all points of a cylindric surface having the same axis as the given cylinder, and is directed normally inwards. Let such a cylinder of radius r be cut by two planes at unit distance apart, perpendicular to the axis, and let us take $\int NdA$ over the surface of the right cylinder of unit length thus enclosed. The value of N at any point of either end of this cylinder is zero, since the whole force is tangential; while at any point of the convex surface the force is normal.

Hence if F denotes the attraction upon unit mass at distance r from the axis, we have

$$\int NdA = F \times \text{convex surface} = 2\pi rF.$$

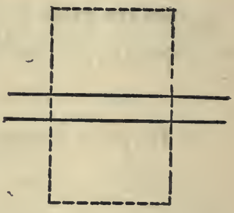
Equating this to $-4\pi M'$, we have for the attraction on unit mass at distance r, using the astronomic unit of mass (page 48, Vol. II, *Statics*),

$$F = -\frac{2M'}{r},$$

where M' is the mass of unit length of the given cylinder, if r is greater than the external radius R of the cylinder. If the cylinder is hollow and r less than the internal radius, M' is zero.

(3) *To find the attraction of a uniform lamina formed by two parallel planes extending to an indefinite distance in all directions.*

Ans. By symmetry the attraction is normal to the lamina, and the same for all points equidistant, whether on the same or on opposite sides.



Consider a right cylinder with plane ends of unit area, these latter being parallel to the lamina and at equal distances on opposite sides, and take $\int NdA$ over its surface. N will be zero over the convex surface because the attraction there is tangential. On the plane ends the attraction will have a uniform value which we will call F. Then $\int NdA = 2F$, and equating this to $-4\pi M'$, we have for the attraction on unit mass, using the astronomic unit of mass (page 48, Vol. II., *Statics*),

$$F = -2\pi M',$$

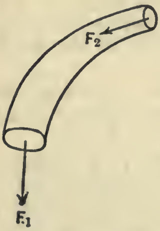
where M' is the mass of unit area of the lamina.

(4) *The force at any point of a tube of force varies inversely as the normal cross-section at that point when there is no attracting matter within the tube.*

Ans. If lines of force are drawn from points in the boundary of any portion of an equipotential surface, the space thus marked off is called a tube of force. Any normal section is then an equipotential surface. If we apply to a tube of force bounded by two normal sections the theorem

$$\int NdA = -4\pi M' \text{ or } 0,$$

N will vanish over the sides of the tube, because the force there is tangential.



At one end (that for which V is greatest) N will be positive and equal to the resultant force F_1; at the other end it will be negative and equal to F_2, taking forces outward as positive and inward negative. Hence denoting the areas of the normal sections by dA_1 and dA_2, we have

$$F_1dA_1 - F_2dA_2 = -4\pi M',$$

where M' denotes the mass of attracting matter contained within the tube.

When there is no attracting matter within the tube

$$F_1dA_1 - F_2dA_2 = 0, \quad \text{or} \quad F_1dA_1 = F_2dA_2;$$

that is, *the force varies inversely as the cross-section of the tube.*

When a liquid flows through a tube, keeping it always full, the flow of liquid, or the volume that passes per unit of time, must be the same for all sections of the tube. If dA be an element of any section and N the component velocity normal to dA, the flow across the section will be $\int NdA$.

If we consider any closed surface in the liquid, the flow into it must equal the flow out of it, and therefore if N is taken positive or negative according as the flow is outward or inward, $\int NdA$ over the whole surface is zero.

From this analogy, NdA, where N denotes the intensity of the component force normal to the element dA, is sometimes called the *flow of force* across the element. We can then say that the *flow of force* is the same across any two sections of a tube of force not separated by attracting matter, that the total flow of force into a space not containing attracting matter is zero, and that the total flow of force into a space containing the quantity of matter M' is $-4\pi M'$.

Faraday used the expression "number of lines of force" to denote what has been called flow of force. It is sometimes called "number of unit tubes of force." Hence the expressions "number of unit tubes of force which cut a surface," the "flow of force across a surface" and the "integral of normal attraction over a surface" are all various names for the integral $\int NdA$.

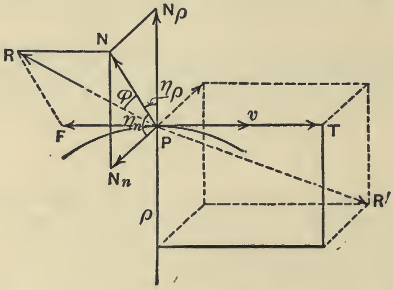
CHAPTER IX.

CONSTRAINED MOTION OF A PARTICLE.

REACTION OF ANY CURVE OR SURFACE. REACTION, EXTERNAL FORCES ZERO. REACTION, CO-PLANAR FORCES. REACTION, PLANE CURVE IN GENERAL. REACTION, PLANE CURVE, CO-PLANAR FORCES. REACTION DUE TO ROTATION OF PATH. CONSTRAINED MOTION, GENERAL EQUATIONS.

Reaction of any Curve or Surface.—Suppose a body sliding upon a rough curve or surface. We can replace it (page 66) by a particle of equal mass m at any point of contact P given by the co-ordinates x, y, z. Let this particle have the velocity v in the direction PT along the tangent at P. Let ρ be the radius of curvature at the point P.

Let R be the reaction of the curve making the angle of kinetic friction ϕ with the normal reaction N. We can resolve R then into the normal reaction N and the friction



$$F = \mu N,$$

where μ is the coefficient of kinetic friction. The friction F always acts opposite to the direction of motion.

Let the normal reaction N make the angles η_ρ and η_n with the radius of curvature ρ and the perpendicular through P to the plane of ρ and PT. Then we can resolve the normal reaction N into the reaction N_ρ along the radius of curvature and the reaction N_n at right angles to the plane of ρ and PT.

If the particle moves on a surface there can be no reaction N_n at right angles to the direction of motion, hence $N_n = 0$. For a surface, then, the resultant R' of all the external forces acting on the particle must always lie in the plane of T and ρ, and the normal reaction $N = N_\rho$ *must always act along the radius of curvature.*

The same will hold true for a curve unless the particle is a ring with the curve passing through it, or the curve is a hollow tube with the particle within it.

Let all the forces and reactions upon the body, except the friction and reaction R at the point P, be F_1, F_2, etc., making with the co-ordinate axes the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc. Then the resultant components F_x, F_y, F_z parallel to the axes are given by

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots = \Sigma F \cos \alpha; \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \ldots = \Sigma F \cos \beta; \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \ldots = \Sigma F \cos \gamma. \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

In taking these algebraic sums, components in the positive directions of x, y and z are positive, in the opposite directions negative.

Let the radius of curvature ρ make the angles θ_x, θ_y, θ_z with the co-ordinate axes. Then we have for the component of the resultant R' of all the external forces F_1, F_2, etc., along the radius of curvature

$$F_x \cos \theta_x + F_y \cos \theta_y + F_z \cos \theta_z.$$

The reaction of the curve or surface due to these external forces is equal and opposite in direction and therefore given by

$$-(F_x \cos \theta_x + F_y \cos \theta_y + F_z \cos \theta_z).$$

But we have seen (page 15) that the reaction of the curve due to motion on the curve alone is $\frac{mv^2}{\rho}$, always acting towards the centre of curvature. The total reaction N_ρ along the radius of curvature is then

$$N_\rho = -(F_x \cos \theta_x + F_y \cos \theta_y + F_z \cos \theta_z) + \frac{mv^2}{\rho}. \quad . \quad . \quad (2)$$

In equation (2), $\frac{mv^2}{\rho}$ always acts towards the centre of curvature and is therefore *always essentially negative.* We take ρ, then, always positive or away from the centre of curvature. We take the components $F_x \cos \theta_x$, etc., positive when acting away from, negative when acting towards, the centre of curvature. If N_ρ comes out positive, then it acts away from, if negative, towards, the centre of curvature. For a particle on the concave side N_ρ negative indicates pressure and positive N_ρ indicates tension between the particle and curve, and *vcie versa* for a particle on the convex side. In any case, then, we have pressure when N_ρ acting upon the particle is *away from the curve or surface.*

If we are dealing with a surface, equation (2) gives the resultant normal reaction N (figure, page 118). So also for a curve, unless the particle is a ring about the curve, or the curve is a hollow tube with the particle inside. Only in this case can we have the normal reaction N_n.

Let the normal at P to the plane of ρ and T make the angles ϵ_x, ϵ_y, ϵ_z with the axes. Then the component of the resultant R' of all the external forces F_1, F_2, etc., along this normal is

$$F_x \cos \epsilon_x + F_y \cos \epsilon_y + F_z \cos \epsilon_z.$$

The reaction N_n of the curve along this normal is equal and opposite, or

$$N_n = -(F_x \cos \epsilon_x + F_y \cos \epsilon_y + F_z \cos \epsilon_z). \quad . \quad . \quad . \quad (3)$$

In equation (3) we take components in any one direction along the normal positive, in the opposite direction negative.

The resultant normal reaction N is then given by

$$N = \sqrt{N_\rho^2 + N_n^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

making the angles η_ρ, η_n with the radius of curvature and the normal to the plane of ρ and PT given by

$$\cos \eta_\rho = \frac{N_\rho}{N}, \quad \cos \eta_n = \frac{N_n}{N}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and angles η_x, η_y, η_z with the axes given by

$$\left.\begin{aligned}\cos \eta_x &= \cos \eta_\rho \cos \theta_x + \cos \eta_n \cos \epsilon_x ;\\ \cos \eta_y &= \cos \eta_\rho \cos \theta_y + \cos \eta_n \cos \epsilon_y ;\\ \cos \eta_z &= \cos \eta_\rho \cos \theta_z + \cos \eta_n \cos \epsilon_z.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

Let the tangent T make the angles ψ_x, ψ_y, ψ_z with the axes. Then we have for the tangential component of the external forces

$$T = F_x \cos \psi_x + F_y \cos \psi_y + F_z \cos \psi_z. \quad . \quad . \quad . \quad . \quad (7)$$

In equation (7) we take components in the direction of motion positive, in the opposite direction negative.

If there is friction there must always be pressure between the particle and curve or surface, or the reaction on the particle must always be away from the curve or surface.

We have then for the friction, when there is any,

$$F = \mu N,$$

where μ is the coefficient of kinetic friction. The direction of the friction is always opposite to the direction of the motion. The resultant tangential force is then

$$T - F = T - \mu N,$$

or

$$T - F = F_x \cos \psi_x + F_y \cos \psi_y + F_z \cos \psi_z - \mu N. \quad . \quad . \quad (8)$$

The resultant reaction R (figure, page 118) lies in the plane of N and T and makes the angle of friction ϕ with N, so that

$$R = \sqrt{N^2 + \mu^2 N^2} = N\sqrt{1 + \mu^2} = \frac{N}{\cos \phi}. \quad . \quad . \quad . \quad . \quad (9)$$

If $T - F = 0$, we have equilibrium. If there is no equilibrium, $T - F$ must be greater or less than zero, and hence

$$\frac{T}{N} \gtrless \mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For a smooth curve or surface $\mu = 0$, $F = 0$. For a straight line or plane $\rho = \infty$ and $\frac{mv^2}{\rho} = 0$.

Reaction—External Forces Zero. — If there are no external forces we have $F_x = 0$, $F_y = 0$, $F_z = 0$. Hence from (3) $N_n = 0$, and from (4) and (5) $N = N_\rho$ and $\eta_\rho = 0$, $\eta_n = 90°$, or the normal reaction lies in the radius of curvature. We have then from (2)

$$N_\rho = + \frac{mv^2}{\rho},$$

always acting towards the centre of curvature. There can only be friction when there is pressure between the curve or surface and particle, that is, when N_ρ is away from the curve, or the particle is *on the concave side.*

Reaction—All Forces Co-planar.—If all the forces are co-planar, R' (figure, page 118) must lie in the plane of ρ and T, and hence $N_n = 0$ and $N = N_\rho$, or the normal reaction lies in the radius of curvature.

Reaction—Plane Curve in General.—For a plane curve in general we may take the plane of the curve that of XY; we have then

$$\theta_z = 90°, \quad \psi_z = 90°, \quad \epsilon_x = 90°, \quad \epsilon_y = 90°, \quad \epsilon_z = 0. \quad . \quad . \quad (1)$$

Hence from (2), page 119,

$$N_\rho = -(F_x \cos\theta_x + F_y \cos\theta_y) + \frac{mv^2}{\rho}. \quad . \quad . \quad . \quad . \quad (2)$$

From (3), page 120, we have

$$N_n = -F_z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and from (4), page 120,

$$N = \sqrt{N_\rho^2 + N_n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (5), page 120, we have then

$$\cos\eta_\rho = \frac{N_\rho}{N}, \quad \cos\eta_n = \frac{N_n}{N}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and from (6),

$$\cos\eta_x = \cos\eta_\rho \cos\theta_x, \quad \cos\eta_y = \cos\eta_\rho \cos\theta_y, \quad \cos\eta_z = \cos\eta_n. \quad (6)$$

In order that there may be friction there must be pressure between the curve and particle or N must act on the particle away from the curve. We have then from (7), page 120,

$$T = F_x \cos\psi_x + F_y \cos\psi_y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the friction

$$F = \mu N,$$

always acting opposite to the direction of motion.

The tangential force is then

$$T - F = F_x \cos\psi_x + F_y \cos\psi_y - \mu N. \quad . \quad . \quad . \quad . \quad (8)$$

If $T - F = 0$, there is equilibrium. If there is no equilibrium, $T - F$ is greater or less than zero, or

$$\frac{T}{N} \gtrless \mu.$$

If the curve is a straight line $\rho = \infty$, $\frac{mv^2}{\rho} = 0$. For a smooth curve $\mu = 0$, $F = 0$.

Reaction—Plane Curve—Co-planar Forces.—For a plane curve when all the forces are in the plane of the curve, we have

$$F_z = 0, \quad \text{and} \quad N_n = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The preceding equation then becomes

$$N = N_\rho = -(F_x \cos \theta_x + F_y \cos \theta_y) + \frac{mv^2}{\rho}. \quad . \quad . \quad . \quad (2)$$

We have also

$$\eta_\rho = 0, \qquad \eta_n = 90;$$

$$\cos \eta_x = \cos \theta_x, \qquad \cos \eta_y = \cos \theta_y, \qquad \cos \eta_z = 0;$$

$$\cos \psi_x = \sin \theta_x, \qquad \cos \psi_y = \cos \theta_x, \qquad \cos \psi_z = 0;$$

$$T = F_x \sin \theta_x + F_y \cos \theta_x; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$F = \mu N_\rho;$$

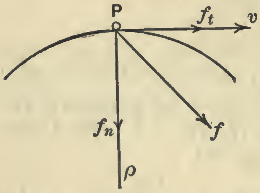
$$T - F = (F_x \sin \theta_x + F_y \cos \theta_x) - \mu N_\rho. \quad . \quad . \quad . \quad . \quad (4)$$

If there is no equilibrium,

$$\frac{T}{N} \gtrless \mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the curve is a straight line, $\rho = \infty$, $\frac{mv^2}{\rho} = 0$. For a smooth curve $\mu = 0$, $F = 0$.

Reaction Due to Rotation of the Path.—If a particle of mass m moves on any curve, it has in general two accelerations with reference to the curve, one, f_t, tangent to the curve and one, $f_n = \frac{v^2}{\rho}$, directed towards the centre of curvature, where v is the velocity and ρ the radius of curvature.



The reaction along the radius of curvature due to motion in the curve alone is then $\frac{mv^2}{\rho}$. The actual acceleration f of the particle is the resultant of f_t and f_n, or

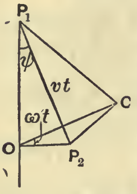
$$f = \sqrt{f_t^2 + f_n^2},$$

and it makes the angles a and b with f_t and f_n given by

$$\cos a = \frac{f_t}{f}, \quad \cos b = \frac{f_n}{f}.$$

The same holds true if curve and particle have any common motion of translation with or without acceleration.

If, however, the curve rotates about an axis we have a relative acceleration due to rotation, besides the relative accelerations f_t and f_n.



Thus, if v is the relative velocity in the curve, $P_1P_2 = vt$ is the distance along the curve described by the point *in an indefinitely small time t.* Let the curve rotate with the angular velocity ω about an axis parallel to $P'O$ which makes the angle ψ with the element of the relative path P_1P_1. Then the angle $P_2OC = \omega t$, and while the particle moves to P_2 the curve moves to P_1C. If then f_r is the acceleration due to rotation of the particle with reference to the curve, we have

$$P_2C = \frac{1}{2}f_rt^2 = vt \sin\psi \,.\, \omega t, \quad \text{or} \quad f_r = -2\omega v \sin\psi. \quad . \quad . \quad . \quad (1)$$

Equation (1) gives the acceleration of the particle with reference to the curve, and it acts in the direction CC_2 in the figure, opposite to the direction of rotation at right angles to the plane of the axis and element of the path.

If the particle is constrained to remain on the curve, the *reaction* of the curve will evidently be in the opposite direction, C_2C, or in the direction of rotation at right angles to the plane of the axis and element of the path.

We have then for the reaction of the curve due to rotation

$$mf_r = 2m\omega v \sin\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

acting in the direction of rotation at right angles to the plane of the axis and element of the path.

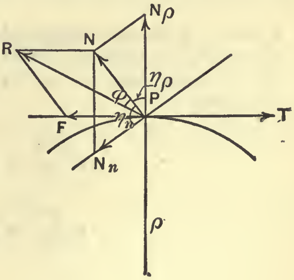
The reaction of the curve due to rotation, then, *is equal to twice the product of the mass and relative velocity v, the angular velocity ω and the sine of the angle ψ which the element of the relative path makes with the line through the particle parallel to the axis. Its direction is at right angles to the plane of this line and element, and it acts in the direction given by the rotation.* (Compare page 216, Vol. I, *Kinematics;* also page 24.)

[**Constrained Motion—General Equations.***—Suppose a body sliding on a rough curve or surface. We can replace it (page 66) by a particle of equal mass m at any point of contact P, given by the co-ordinates x, y, z. Let the particle have the velocity v in the direction PT along the tangent at P. Let ρ be the radius of curvature at the point P.

Let R be the reaction of the curve making the angle of kinetic friction ϕ with the normal reaction N. We can resolve R into the normal reaction N and the friction

$$F = \mu N,$$

where μ is the coefficient of kinetic friction. The friction F always acts opposite to the direction of motion.



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* Students not familiar with the Calculus should omit the rest of this chapter.

Let the normal reaction N make the angles η_ρ and η_n with the radius of curvature ρ and the perpendicular through P to the plane of ρ and PT, and the angles η_x, η_y, η_z with the axes.

Let N_ρ be the component of the normal reaction N along the radius of curvature, and let the radius of curvature make the angles θ_x, θ_y, θ_z with the axes.

Let the direction of motion PT make the angles ψ_x, ψ_y, ψ_z with the axes, so that

$$\cos \psi_x = \frac{dx}{ds}, \quad \cos \psi_y = \frac{dy}{ds}, \quad \cos \psi_z = \frac{dz}{ds}.$$

Let N_n be the component of the normal reaction N at right angles to the plane of ρ and PT, making the angles ϵ_x, ϵ_y, ϵ_z with the axes.

If the particle moves on a surface there can be no reaction N_n at right angles to the direction of motion. For a surface, then, the normal reaction N must act along the radius of curvature and $N = N_\rho$.

The same holds true for a curve unless the particle is a ring with the curve passing through it, or the curve is a hollow tube with the particle inside.

Let all the forces and reactions upon the body, except the friction and reaction at the point P, be F_1, F_2, etc., making with the axes the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc.

Then the resultant components F_x, F_y, F_z parallel to the axes are given by

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots = \Sigma F \cos \alpha; \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \ldots = \Sigma F \cos \beta; \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \ldots = \Sigma F \cos \gamma. \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

The same conventions as to signs hold as in equations (1), page 119.

If these forces alone act, we have unconstrained motion.

Motion on a Curve.—If there is friction, there must always be pressure between the curve and particle, or N must always act on the particle away from the curve. The friction is then

$$F = \mu N,$$

where μ is the coefficient of kinetic friction. This friction is always opposite to the direction of motion, or is a retarding force.

We have then for the components along the axes of the forces acting on the particle

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= F_x + N \cos \eta_x - \mu N \frac{dx}{ds}; \\ m\frac{d^2y}{dt^2} &= F_y + N \cos \eta_y - \mu N \frac{dy}{ds}; \\ m\frac{d^2z}{dt^2} &= F_z + N \cos \eta_z - \mu N \frac{dz}{ds}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If N is zero, the motion is unconstrained. Equations (2), together with the equation of the curve, determine the motion.

Since the particle is constrained to move on the curve, the motion along any normal is zero. We have then as conditions of constraint

$$\left.\begin{aligned} dx\cos\eta_x + dy\cos\eta_y + dz\cos\eta_z &= 0;\\ dx\cos\theta_x + dy\cos\theta_y + dz\cos\theta_z &= 0;\\ dx\cos\epsilon_x + dy\cos\epsilon_y + dz\cos\epsilon_z &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Also, since the resultant force at right angles to the plane of ρ and PT is zero and the force along the radius of curvature always acting towards the centre of curvature is $\frac{mv^2}{\rho}$, we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2}\cos\epsilon_x + \frac{d^2y}{dt^2}\cos\epsilon_y + \frac{d^2z}{dt^2}\cos\epsilon_z &= 0;\\ \frac{d^2x}{dt^2}\cos\theta_x + \frac{d^2y}{dt^2}\cos\theta_y + \frac{d^2z}{dt^2}\cos\theta_z &= \frac{v^2}{\rho}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

If we multiply equations (2) severally by $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ and add, we have, since

$$N_\rho = N\cos\eta_x\cos\theta_x + N\cos\eta_y\cos\theta_y + N\cos\eta_z\cos\theta_z,$$

after reducing by (3) and (4),

$$N_\rho = -(F_x\cos\theta_x + F_y\cos\theta_y + F_z\cos\theta_z) + \frac{mv^2}{\rho}. \quad . \quad . \quad . \quad (5)$$

This is the same as equation (2), page 119, and the conventions for signs are the same as there indicated.

If we multiply equations (2) severally by $\cos\epsilon_x$, $\cos\epsilon_y$, $\cos\epsilon_z$ and add, we have, since

$$N_n = N\cos\eta_x\cos\epsilon_x + N\cos\eta_y\cos\epsilon_y + N\cos\eta_z\cos\epsilon_z,$$

after reducing by (3) and (4),

$$N_n = -(F_x\cos\epsilon_x + F_y\cos\epsilon_y + F_z\cos\epsilon_z). \quad . \quad . \quad . \quad (6)$$

This is the same as equation (3), page 120, and the conventions for signs are the same as there indicated.

We have then, as on page 120, the resultant normal reaction

$$N = \sqrt{N_\rho^2 + N_n^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

making the angles η_ρ, η_n with the radius of curvature and the normal to the plane of ρ and PT given by

$$\cos\eta_\rho = \frac{N_\rho}{N}, \qquad \cos\eta_n = \frac{N_n}{N}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and angles η_x, η_y, η_z with the axes given by

$$\left.\begin{aligned} \cos\eta_x &= \cos\eta_\rho\cos\theta_x + \cos\eta_n\cos\epsilon_x;\\ \cos\eta_y &= \cos\eta_\rho\cos\theta_y + \cos\eta_n\cos\epsilon_y;\\ \cos\eta_z &= \cos\eta_\rho\cos\theta_z + \cos\eta_n\cos\epsilon_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We have also

$$\cos\theta_x = \rho\frac{d}{ds}\left(\frac{dx}{ds}\right), \quad \cos\theta_y = \rho\frac{d}{ds}\left(\frac{dy}{ds}\right), \quad \cos\theta_z = \rho\frac{d}{ds}\left(\frac{dz}{ds}\right), \quad . \quad (10)$$

where ρ is always to be taken away from the centre of curvature or positive, and is given by

$$\rho = \pm\frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}}.$$

If we multiply equations (2) severally by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ and add, we have, since $dx^2 + dy^2 + dz^2 = ds^2$, after reducing by (3), remembering that

$$m\left(\frac{d^2x}{dt^2}\cdot\frac{dx}{ds} + \frac{d^2y}{dt^2}\cdot\frac{dy}{ds} + \frac{d^2z}{dt^2}\cdot\frac{dz}{ds}\right) = m\frac{dv}{dt}$$

for the resultant tangential force

$$m\frac{dv}{dt} = F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds} - \mu N. \quad . \quad . \quad . \quad . \quad (11)$$

If T is the tangential component of the external forces we have

$$m\frac{dv}{dt} = T - \mu N$$

and

$$T = F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is equation (7), page 120.

The resultant reaction R lies in the plane of N and T, and makes the angle of friction ϕ with N so that

$$R = \sqrt{N^2 + \mu^2N^2} = N\sqrt{1 + \mu^2} = \frac{N}{\cos\phi}. \quad . \quad . \quad . \quad . \quad (13)$$

Again, if we multiply equations (2) severally by dx, dy, dz and add, we have, after reduction by (3), for the differential *work*, $m\frac{dv}{dt}\cdot ds$, since

$$\frac{d^2x\,dx + d^2y\,dy + d^2z\,dz}{dt^2} = \frac{1}{2}d\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right) = \frac{1}{2}d\left(\frac{ds^2}{dt^2}\right) = \frac{1}{2}dv^2 = v\,dv,$$

$$mv\,dv = F_xdx + F_ydy + F_zdz - \mu Nds = (T - \mu N)ds. \quad . \quad . \quad (14)$$

If we integrate (14) and let $v = v_1$ when $t = 0$, we have for the work

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = \int_0 (T-\mu N)ds = \int_0^t (F_xdx+F_ydy+F_zdz-\mu Nds). \quad (15)$$

That is, the gain of kinetic energy is equal to the work done (page 87). Equation (15) gives the velocity v.

Since $v = \frac{ds}{dt}$, we have from (15)

$$dt = \frac{ds}{\left[v_1^2 + \frac{2}{m}\int_0^t (T - \mu N)ds\right]}$$

$$= \frac{ds}{\left[v_1^2 + \frac{2}{m}\int_0^t (F_x dx + F_y dy + F_z dz - \mu N ds)\right]^{\frac{1}{2}}}. \quad . \quad . \quad (16)$$

Plane Curve in General.—If the curve is a plane curve we may take the plane of the curve that of XY. We have then $dz = 0$, $\cos\theta_x = \frac{dy}{ds}$, $\cos\theta_y = \frac{dx}{ds}$, $\cos\theta_z = 0$, $\cos\epsilon_x = 0$, $\cos\epsilon_y = 0$, $\cos\epsilon_z = 1$. Hence

$$N_\rho = -\left(F_x\frac{dy}{dx} + F_y\frac{dx}{ds}\right) + \frac{mv^2}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ρ is always positive;

$$N_n = -F_z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$N = \sqrt{N_\rho^2 + N_n^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\cos\eta_\rho = \frac{N_\rho}{N}, \quad \cos\eta_n = \frac{N_n}{N}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\cos\eta_x = \cos\eta_\rho\frac{dy}{ds}, \quad \cos\eta_y = \cos\eta_\rho\frac{dx}{ds}, \quad \cos\eta_z = \cos\eta_n; \quad . \quad (5)$$

$$T = F_x\frac{dx}{ds} + F_y\frac{dy}{ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The friction F is given by

$$F = \mu N. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7$$

The resultant tangential force is

$$m\frac{dv}{dt} = T - F = F_x\frac{dx}{ds} + F_y\frac{dy}{ds} - \mu N. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The work is

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = \int_0^t (F_x dx + F_y dy - \mu N ds). \quad . \quad . \quad . \quad (9)$$

We have from (9)

$$dt = \frac{ds}{\left[v_1^2 + \frac{2}{m}\int_0^t (F_x dx + F_y dy - \mu N ds)\right]^{\frac{1}{2}}}. \quad . \quad . \quad . \quad (10)$$

If all the forces are in the plane of the curve we have in these equations $F_z = 0$, $N_n = 0$, $N = N_\rho$, $\cos \eta_\rho = 1$, $\cos \eta_n = 0$.

2. Motion on a Surface.—Let the equation of the surface be

$$u = 0,$$

where u is a function of x, y; z. Let

$$U = \frac{du}{dx}, \quad V = \frac{du}{dy}, \quad W = \frac{du}{dz}, \quad \text{and} \quad U^2 + V^2 + W^2 = Q^2.$$

The normal reaction for a surface is, as we have seen (page 118), always along the radius of curvature, or $N = N_\rho$. Let the radius of curvature make the angles θ_x, θ_y, θ_z with the axes. Then its direction-cosines are

$$\cos \theta_x = \frac{U}{Q}, \quad \cos \theta_y = \frac{V}{Q}, \quad \cos \theta_z = \frac{W}{Q}. \quad . \quad . \quad . \quad . \quad (1)$$

If, then, in equations (2), page 124, we put θ in place of η, we have for the resultant components parallel to the axes of all the forces acting upon the particle

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= F_x + N\frac{U}{Q} - \mu N\frac{dx}{ds}; \\ m\frac{d^2y}{dt^2} &= F_y + N'\frac{V}{Q} - \mu N\frac{dy}{ds}; \\ m\frac{d^2z}{dt^2} &= F_z + N'\frac{W}{Q} - \mu N\frac{dz}{ds}; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Proceeding then just as before, we find

$$N = N_\rho = -\left(\frac{F_x U + F_y V + F_z W}{Q}\right) + \frac{mv^2}{\rho}, \quad . \quad . \quad . \quad . \quad (3)$$

with the same conventions as to signs as in equation (5), page 125. If N_ρ comes out negative with reference to the surface there is no friction.

We also have

$$T = F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\frac{mdv}{dt} = T - F = F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds} - \mu N, \quad . \quad . \quad . \quad . \quad (5)$$

with the same conventions as to sign as in equations (11), (12), page 126.

We also have

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = \int_{t=0}^{t}(T - \mu N)ds = \int_{t=0}^{t}(F_x dx + F_y dy + F_z dz - \mu N ds), \quad (6)$$

and

$$dt = \frac{ds}{\left[v_1^2 + \frac{2}{m}\int_{t=0}^{t}(T - \mu N)ds\right]^{\frac{1}{2}}}$$

$$= \frac{ds}{\left[v_1^2 + \frac{2}{m}\int_{t=0}^{t}(F_x dx + F_y dy + F_z dz - \mu N ds)\right]^{\frac{1}{2}}}, \quad . \quad . \quad (7)$$

with the same conventions as to sign as in equations (15), (16), page 127. If the surface is smooth, $\mu = 0$ in all equations.

COR. If the surface is smooth and there are no external forces we have $\mu = 0$, $F_x = 0$, $F_y = 0$, $F_z = 0$. In this case the only force acting upon the particle is $N_\rho = \frac{mv^2}{\rho}$ always acting towards the centre of curvature along the radius of curvature and therefore always at right angles to the direction of motion. There is then no change of speed and v is constant in magnitude at every point of the path. There is then no change of kinetic energy, and hence no work is done by or against the normal reaction N_ρ. *The radius of curvature of the path on the surface must therefore be constant.*

Such a line on a surface is called a **geodesic line.**

From equations (2), page 124, we have in this case

$$\frac{m\frac{d^2x}{dt^2}}{U} = \frac{m\frac{d^2y}{dt^2}}{V} = \frac{m\frac{d^2z}{dt^2}}{W} = \frac{mv^2}{\rho Q}.$$

Dividing by $v^2 = \frac{ds^2}{dt^2}$, we have

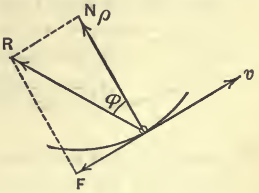
$$\frac{\frac{d^2x}{ds^2}}{U} = \frac{\frac{d^2y}{ds^2}}{V} = \frac{\frac{d^2z}{ds^2}}{W} = \frac{1}{\rho Q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equations (8) are then the equations of a geodesic line on a surface.

EXAMPLES.

(1) *Find the motion of a particle on a curve under the action of friction and the curve reaction only.*

[Ans. In this case we have the external forces zero, or $F_y = 0$, $F_x = 0$, $F_z = 0$. Hence $N_n = 0$, $N = N_\rho = \frac{mv^2}{\rho}$, or the normal reaction is always along the radius of curvature and acts towards the centre of curvature. In order that there may be friction N_ρ must always be positive with reference to the curve, hence the particle must be on the concave side of the curve.



The friction is then

$$F = \mu N_\rho = \mu\frac{mv^2}{\rho},$$

acting always opposite to the direction of motion.

The reaction R makes an angle with the normal always equal to the angle of friction ϕ, so that $\mu = \tan\phi$.

We have then

$$R = \frac{mv^2}{\rho}\sqrt{1+\mu^2} = \frac{mv^2}{\rho\cos\phi}.$$

The tangential force is

$$\frac{mdv}{dt} = -\mu\frac{mv^2}{\rho}, \quad\text{or}\quad \frac{dv}{dt} = -\mu\frac{v^2}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or the motion is retarded.

From (1) we have

$$\frac{dv}{v^2} = -\mu\frac{dt}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, since $vdt = ds$,

$$\frac{dv}{v} = -\mu\frac{ds}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating (2), we have, if v_1 is the initial velocity when $t = 0$ and $s = 0$,

$$\frac{1}{v} - \frac{1}{v_1} = \mu\int_{t=0}^{t}\frac{dt}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Integrating (3), we have

$$\operatorname{logn}\frac{v}{v_1} = -\mu\int_0^s\frac{ds}{\rho}, \quad\text{or}\quad v = v_1 e^{-\mu\int_0^s\frac{ds}{\rho}}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But $\frac{ds}{\rho} = d\theta =$ the angle between two successive tangents, hence if θ is the angle between the tangents for the initial and final positions,

$$v = v_1 e^{-\mu\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $e = 2.718282 =$ base of Naperian system of logarithms.

If the curve is a circle ρ is constant and equal to the radius r, and we have from (4) and (5)

$$v = \frac{v_1}{1+\frac{\mu v_1}{r}t}, \quad v = v_1 e^{-\frac{\mu s}{r}} = v_1 e^{-\mu\theta},$$

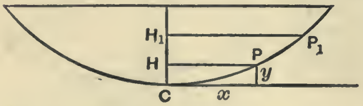
and

$$s = \frac{r}{\mu}\operatorname{logn}\left(1+\frac{\mu v_1}{r}t\right),$$

$$\theta = \frac{1}{\mu}\operatorname{logn}\left(1+\frac{\mu v_1}{r}t\right).$$

(2) *Find the motion of a particle on a cycloid, neglecting friction.*

Ans. Let the initial position be P_1 at the height $CH_1 = h$ above C, and the speed at P_1 be zero, and at P be v. We have then the gain of kinetic equal to the loss of potential energy, or



$$\frac{1}{2}mv^2 = mg(h-y); \quad \therefore\ v = \sqrt{2g(h-y)}.$$

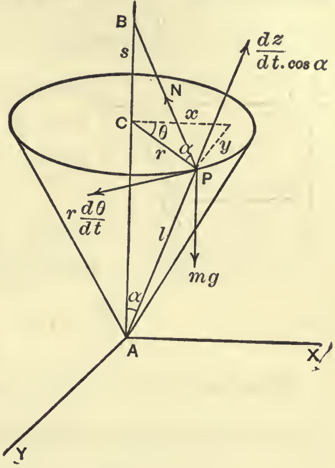
We can then find the motion as on page 157, Vol. I, *Kinematics*.

(3) *Find the motion of a particle on a circle, neglecting friction.*

Ans. We have as before $v = \sqrt{2g(h-y)}$. We can then find the motion as on page 159, Vol. I, *Kinematics*.

(4) *Find the motion of a particle on a conical surface, neglecting friction.*

[Ans. Let the angle of the cone be α and the axis vertical. Let the particle be at P and have the velocity v in any direction, and let its distance AP from the vertex be l, and the distance CP from the axis be r. Take A as origin and let the angle of r with the axis of X be θ.



Then we have

$$x^2 + y^2 + z^2 = l^2,$$

or, since $l = \dfrac{z}{\cos\alpha}$,

$$u = x + y^2 + z^2 - \frac{z^2}{\cos^2\alpha} = 0.$$

Hence (page 128)

$$U = \frac{du}{dx} = 2x, \quad V = \frac{du}{dy} = 2y,$$

$$W = \frac{du}{dz} = 2z\left(1 - \frac{1}{\cos^2\alpha}\right). \quad . \qquad (1)$$

or, since $z\tan\alpha = r$ and $z\left(1 - \dfrac{1}{\cos^2\alpha}\right) = -z\tan^2\alpha$,

$$W = -2r\tan\alpha.$$

Hence

$$Q^2 = U^2 + V^2 + W^2 = 4r^2(1 + \tan^2\alpha) = \frac{4r^2}{\cos^2\alpha},$$

or

$$Q = \frac{2r}{\cos\alpha}.$$

We have then from equations (2), page 128, since $\mu = 0$, $F_x = 0$, $F_y = 0$, $F_z = -mg$,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{N\cos\alpha}{m}\cdot\frac{x}{r}; \\ \frac{d^2y}{dt^2} &= \frac{N\cos\alpha}{m}\cdot\frac{y}{r}; \\ \frac{d^2z}{dt^2} &= -\frac{N\sin\alpha}{m} - g. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (2)$$

We have also from (3), page 128, since $\rho = \dfrac{r}{\cos\alpha}$, and $\dfrac{mv^2}{\rho}$ is always towards the centre of curvature or negative in direction,

$$N = -mg\sin\alpha - \frac{mv^2\cos\alpha}{r}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (3)$$

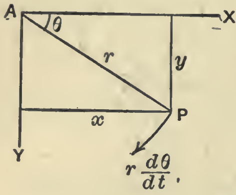
From equation (6), page 128, we have for $\mu = 0$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = \int_0^t - mgdz = mg(z_1 - z), \quad . \; . \; . \; . \qquad (4)$$

where z_1 is the ordinate for the initial position of P. That is, the gain of kinetic equals the loss of potential energy.

If we multiply the first of equations (2) by y and the second by x and subtract, we have

$$\frac{d^2x}{dt^2}y - \frac{d^2y}{dt^2}x = 0, \quad \text{or} \quad \frac{dx}{dt}y - \frac{dy}{dt}x = \text{Const.} \quad . \quad . \quad . \quad . \quad (5)$$



But equation (5) gives the moment of the horizontal velocity $r\frac{d\theta}{dt}$ with reference to the axis of Z. Hence

$$r^2\frac{d\theta}{dt} = \text{Const.} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we suppose then the initial velocity v_1 *to be horizontal at the distance* r_1, we have from (6)

$$r^2\frac{d\theta}{dt} = r_1v_1, \quad \text{or} \quad r\frac{d\theta}{dt} = v_1\frac{r_1}{r} = v_1\frac{z_1}{z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Equation (7) gives the horizontal velocity $r\frac{d\theta}{dt}$. The vertical velocity is $\frac{dz}{dt}$, and its component along AP (figure, page 131) is $\frac{dz}{dt \,.\, \cos\alpha}$. We have then

$$v^2 = \left(r\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt \,.\, \cos\alpha}\right)^2 = v_1^2\frac{z_1^2}{z^2} + \left(\frac{dz}{dt \,.\, \cos\alpha}\right)^2. \quad . \quad . \quad . \quad (8)$$

If we put this value of v^2 equal to that found in (4), we have

$$2g\left(\frac{v_1^2}{2g} + z_1 - z\right) = v_1^2\frac{z_1^2}{z^2} + \left(\frac{dz}{dt \,.\, \cos\alpha}\right)^2.$$

If we put $h = \frac{v_1^2}{2g}$ = height due to initial velocity, we obtain

$$\left(\frac{dz}{dt}\right)^2 = -\frac{2g\cos^2\alpha}{z^2}[z^3 - (h + z_1)z^2 + hz_1^2],$$

or

$$\left(\frac{dz}{dt}\right)^2 = -\frac{2g\cos^2\alpha}{z^2}(z - z_1)[z^2 - hz - hz_1]. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (9) shows that there are two values of z for which the vertical component of the velocity $\frac{dz}{dt}$ is zero. One is $z = z_1$, the other, z_2, is given by putting the last factor on the right of equation (9) equal to zero. This gives

$$z_2^2 - hz_2 - hz_1 = 0, \quad \text{or} \quad z_2 = \frac{1}{2}h + \sqrt{\left(\frac{1}{2}h\right)^2 + hz_1}. \quad . \quad . \quad . \quad (10)$$

We see from this that

z_2 is greater than z_1 when $h = \frac{v_1^2}{2g} > \frac{1}{2}z_1$, or $v_1 > \sqrt{gz_1}$;

z_2 is equal to z_1 when $h = \frac{v_1^2}{2g} = \frac{1}{2}z_1$, or $v_1 = \sqrt{gz_1}$;

z_2 is less than z_1 when $h = \frac{v_1^2}{2g} < \frac{1}{2}z_1$, or $v_1 < \sqrt{gz_1}$.

In the first case, when $h = \frac{v_1^2}{2g}$ is greater than $\frac{1}{2}z_1$, or the initial horizontal velocity v_1 is greater than $\sqrt{gz_1}$, the particle traces a spiral on the surface and will *rise* through a distance $z_2 - z_1$, to a point where z_2 is given by (10). At this point the particle remains in a horizontal plane and describes continually a horizontal circle of radius $r_2 = z_2 \tan \alpha = z_2 \frac{r_1}{z_1}$ with the constant speed v_1. The periodic time is then

$$t = \frac{2\pi r_2}{v_1} = \frac{2\pi r_1}{v_1 z_1} z_2. \qquad (11)$$

In the second case, when $h = \frac{v_1^2}{2g}$ is equal to $\frac{1}{2}z_1$, or the initial horizontal velocity $v_1 = \sqrt{gz_1}$, the particle remains in the horizontal plane in which it starts. Its periodic time is

$$t = \frac{2\pi r_1}{v_1}. \qquad (12)$$

In the third case, when $h = \frac{v_1^2}{2g}$ is less than $\frac{1}{2}z_1$, or the initial horizontal velocity v_1 is less than $\sqrt{gz_1}$, the particle traces a spiral on the surface and will *fall* through a distance $z_1 - z_2$ to a point where z_2 is given by (10). At this point the particle remains in a horizontal plane and its periodic time is given by (11).

(5) *Find the motion of a particle on a conical surface, neglecting friction and the weight of the particle.*

[Ans. In the preceding example we have only to put $g = 0$. From equation (4), page 131, we see then that the speed does not change, and $v = v_1$. Hence from equation (8), page 132, we have

$$\frac{z\,dz}{\sqrt{z^2 - z_1^2}} = v \cos \alpha \,.\, dt. \qquad (1)$$

Integrating, since when $t = 0$, $z = z_1$, we have

$$\sqrt{z^2 - z_1^2} = vt \cos \alpha. \qquad (2)$$

The greater the time t the more nearly (2) approaches to

$$z = vt \cos \alpha, \quad \text{or} \quad \frac{dz}{dt} = v \cos \alpha.$$

The vertical velocity, then, approaches the limit $v \cos \alpha$.

From equation (7), page 132, we have

$$\frac{d\theta}{dt} = \frac{vz_1}{rz} = \frac{vz_1}{z^2 \tan \alpha}. \qquad (3)$$

Substituting the value of z from (2), we have

$$\sin \alpha \,.\, d\theta = \frac{v}{z_1} \cos \alpha \frac{dt}{1 + \left(\frac{vt}{z_1} \cos \alpha\right)^2}. \qquad (4)$$

Integrating and letting $\theta = \theta_1$ when $t = 0$, we have

$$(\theta - \theta_1) \sin \alpha = \tan^{-1}\left(\frac{v}{z_1} \cos \alpha \,.\, t\right), \qquad (5)$$

or

$$v \cos \alpha \,.\, t = z_1 \tan [(\theta - \theta_1) \sin \alpha]. \qquad (6)$$

If we insert this in (2), we obtain

$$\sqrt{z^2 - z_1^2} = z_1 \tan [(\theta - \theta_1) \sin \alpha], \quad . \ . \ . \ . \ . \ . \ . \ . \quad (7)$$

or

$$\frac{z}{z_1} = \sqrt{1 + \tan^2 [(\theta - \theta_1) \sin \alpha]} = \frac{1}{\cos [(\theta - \theta_1) \sin \alpha]}. \quad . \ . \quad (8)$$

Since $\frac{z}{z_1} = \frac{r}{r_1}$, we have

$$r = \frac{r_1}{\cos [(\theta - \theta_1) \sin \alpha]}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

If we take the co-ordinate axes so that the initial point of the path is in the plane of XZ, we have $\theta_1 = 0$, and

$$r = \frac{r_1}{\cos (\theta \sin \alpha)}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (10)$$

Equation (10) shows that $r = \infty$ when $\theta \sin \alpha = \frac{\pi}{2}$.

The angle $\theta = \frac{\pi}{2 \sin \alpha}$ gives then the position of that element of the conical surface which is an asymptote to the path, or is tangent to the path at an infinite distance from the vertex.

We have then, from (2), for the distance of the point above the vertex at any time t,

$$z = \sqrt{z_1^2 + v^2t^2 \cos \alpha}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (11)$$

The radius at this point is $r = z \tan \alpha$, and from (10) the angle described is given by

$$\cos (\theta \sin \alpha) = \frac{r_1}{r} = \frac{r_1}{z \tan \alpha}. \quad . \ . \ . \ . \ . \ . \ . \quad (12)$$

(6) *Find the motion of a particle on the surface of a sphere, disregarding friction.*

[Ans. Take the origin at the centre of the sphere.
Then we have, if r is the radius,

$$u = x^2 + y^2 + z^2 - r^2 = 0; \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

$$U = \frac{du}{dx} = 2x, \quad V = \frac{du}{dy} = 2y, \quad W = \frac{du}{dz} = 2z;$$

$$Q = 2r.$$

We have then from equations (2), page 128, since $F_x = 0$,

$$F_y = 0, \quad F_z = -mg.$$

$$\frac{d^2x}{d^2t} = \frac{N}{m} \cdot \frac{x}{r}; \qquad \frac{d^2y}{dt^2} = \frac{N}{m} \cdot \frac{y}{r}; \qquad \frac{d^2z}{dt^2} = -g + \frac{N}{m} \cdot \frac{z}{r}. \quad . \ . \quad (2)$$

We have also from equation (3), page 128, since $\frac{mv^2}{r}$ always acts towards the centre and is therefore negative in direction,

$$N = mg\frac{z}{r} - \frac{mv^2}{r}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

From equation (6), page 128, we have

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = mg(z_1 - z), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

where v_1 and z_1 are the initial values of v and z.

If we multiply the first of equations (2) by y, the second by x, and subtract, we have

$$\frac{d^2x}{dt^2}y - \frac{d^2y}{dt^2}x = 0, \quad \text{or} \quad \frac{dx}{dt}y - \frac{dy}{dt}x = \text{Const.} \quad . \; . \; . \; . \quad (5)$$

But equation (5) gives the moment of the horizontal velocity with reference to the axis of Z. If then we take the initial velocity v_1 horizontal at the distance $\sqrt{r^2 - z_1^2}$ from the axis of Z, we have

$$\frac{dx}{dt}y - \frac{dy}{dt}x = v_1\sqrt{r^2 - z_1^2}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

From (1) we have

$$x\frac{dx}{dt} + y\frac{dy}{dt} = -z\frac{dz}{dt}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Squaring (6) and (7) and adding, we have

$$(x^2 + y^2)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = v_1^2(r^2 - z_1^2) + z^2\left(\frac{dz}{dt}\right)^2$$

But $x^2 + y^2 = r^2 - z^2$ and $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v^2 - \left(\frac{dz}{dt}\right)^2$. Hence

$$(r^2 - z^2)\left[v^2 - \left(\frac{dz}{dt}\right)^2\right] = v_1^2(r^2 - z_1^2) + z^2\left(\frac{dz}{dt}\right)^2. \quad . \; . \; . \quad (8)$$

From (8) we have

$$\left(\frac{dz}{dt}\right)^2 = \frac{(r^2 - z^2)v^2 - (r^2 - z_1^2)v_1^2}{r^2},$$

or substituting the value of v^2 from (4) and putting $\frac{v_1^2}{2g} = h$

$$\left(\frac{dz}{dt}\right)^2 = \frac{2g}{r^2}\left[(r^2 - z^2)(h - z + z_1) - h(r^2 - z_1^2)\right],$$

or

$$\frac{r^2}{2g}\left(\frac{dz}{dt}\right)^2 = (z - z_1)[z^2 - hz - (hz_1 + r^2)]. \quad . \; . \; . \; . \; . \; . \quad (9)$$

Equation (9) shows that there are two values of z for which the vertical component of the velocity $\frac{dz}{dt}$ is zero. One is $z = z_1$, the other, z_2, is given by putting the last factor on the right of equation (9) equal to zero. This gives

$$z_2^2 - hz_2 = hz_1 + r^2, \quad \text{or} \quad z_2 = \frac{h}{2} + \sqrt{hz_1 + r^2 + \frac{h^2}{4}}. \quad . \; . \quad (10)$$

We see from this that

z_2 is equal to z_1 when $h = \frac{z_1^2 - r^2}{2z_1}$, or $v_1 = \sqrt{\frac{g(z_1^2 - r^2)}{z_1}}$;

z_2 is greater than z_1 when $h > \frac{z_1^2 - r^2}{2z_1}$, or $v_1 > \sqrt{\frac{g(z_1^2 - r^2)}{z_1}}$;

z_2 is less than z_1 when $h < \frac{z_1^2 - r^2}{2z_1}$, or $v_1 < \sqrt{\frac{g(z_1^2 - r^2)}{z_1}}$.

We see that z_1 must always be negative in order that v_1 may have a real value. Also, in the first case when the initial horizontal velocity

$$v_1 = \sqrt{\frac{g(z_1^2 - r^2)}{z_1}},$$

the particle remains in a horizontal plane and describes continually a horizontal circle of radius $\sqrt{r^2 - z_1^2}$ with the constant speed v_1. The periodic time is then

$$t = \frac{2\pi\sqrt{r^2 - z_1^2}}{v_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If v_1 is greater or less than $\sqrt{\frac{g(z_1^2 - r^2)}{z_1}}$, the particle traces a spiral on the surface and will rise or fall through the distance $z_2 - z_1$ to a point where z_2 is given by (10). At this point the particle remains in a horizontal plane and its periodic time is given by (11).

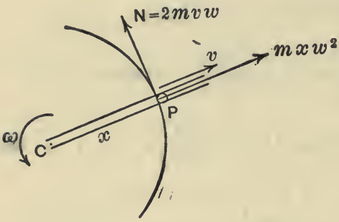
(7) *Find the motion of a particle in a smooth straight tube which revolves uniformly round a vertical axis to which it is perpendicular.*

[Ans. Let x be the distance of the particle P of mass m from the centre of rotation C, let ω be the angular velocity and let the initial velocity along the tube be v_1 away from C. The acceleration of the particle with reference to the centre is away or positive and equal to $x\omega^2$. We have then

$$\frac{d^2x}{dt^2} = +\,\omega^2 x. \quad . \quad . \quad . \quad (1)$$

The general integral of this is

$$x = Ae^{+\omega t} + Be^{-\omega t}. \quad . \quad . \quad (2)$$



Differentiating (2), we have

$$v = \frac{dx}{dt} = A\omega e^{+\omega t} - B\omega e^{-\omega t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where e is the base of the Naperian system of logarithms and A and B are constants of integration. To determine these constants, let $x = x_1$ and $v = v_1$ when $t = 0$. We have then

$$x_1 = A + B \quad \text{and} \quad v_1 = A\omega - B\omega.$$

Hence

$$A = \frac{1}{2}\left(x_1 + \frac{v_1}{\omega}\right), \qquad B = \frac{1}{2}\left(x_1 - \frac{v_1}{\omega}\right).$$

Substituting in (2) and (3), we have

$$x = \frac{1}{2}\left(x_1 + \frac{v_1}{\omega}\right)e^{+\omega t} + \frac{1}{2}\left(x_1 - \frac{v_1}{\omega}\right)e^{-\omega t}; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$v = \frac{1}{2}\left(x_1\omega + v_1\right)e^{+\omega t} - \frac{1}{2}\left(x_1\omega - v_1\right)e^{-\omega t}. \quad . \quad . \quad . \quad . \quad (5)$$

Again, if we multiply both sides of (1) by $2dx$, we have

$$2\frac{dx}{dt} \cdot d\left(\frac{dx}{dt}\right) = 2\omega^2 x\,dx, \quad \text{or} \quad 2v\,dv = 2\omega^2 x\,dx.$$

Integrating, and making $v = v_1$ for $x = x_1$, we have

$$v^2 - v_1^2 = \omega^2 x^2 - \omega^2 x_1^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or the difference of the squares of the velocities in the tube equals the difference of the squares of the velocities of rotation.

From page 123 we have for the reaction of the tube on the particle

$$N = 2mv\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

acting in the direction of rotation.

Substituting the value of v from (5) and (6), we have

$$N = m(x_1\omega^2 + v_1\omega)e^{+\omega t} - m(x_1\omega^2 - v_1\omega)e^{-\omega t}, \quad . \quad . \quad . \quad . \quad (8)$$

or

$$N = 2m\omega\sqrt{v_1^2 + \omega^2(x^2 - x_1^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we make $v_1 = x_1\omega$, we have $v = x\omega$; and if x is the length of the tube, the particle leaves the end of the tube with the absolute velocity $x\omega\sqrt{2}$, at an angle of 45° with the tube, and moves uniformly with that velocity after leaving the tube.

We can also deduce (6) by the principle of kinetic energy as follows:

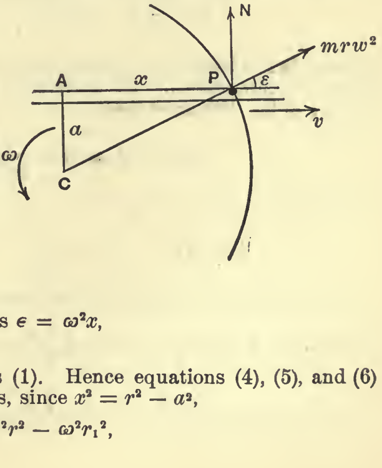
From (7) the average reaction is $\frac{2m(x\omega - x_1\omega)}{t}$, the distance is $\frac{x\omega + x_1\omega}{2}t$. The work done is then $m(\omega^2x^2 - \omega^2x_1^2)$. The initial absolute kinetic energy is $\frac{1}{2}m(v_1^2 + \omega^2x_1^2)$, and the final absolute kinetic energy is $\frac{1}{2}m(v^2+\omega^2x^2)$. Hence, since gain of kinetic energy is equal to work done,

$$\frac{1}{2}m(v^2 + \omega^2x^2) - \frac{1}{2}m(v_1^2 + \omega^2x_1^2) = m(\omega^2x^2 - \omega^2x_1^2),$$

from which we obtain equation (6).

If the initial velocity v_1 is towards the centre of rotation, we have only to take v_1 negative and v negative in the preceding equations. N then is negative, or acts opposite to the direction of rotation. If in this case we suppose $v = 0$ when $x = 0$, we have, from (6), $v_1 = -x_1\omega$. If then the initial velocity towards the centre is equal to the velocity of rotation, the particle will arrive at the centre with a final velocity of zero.

If the centre of rotation is outside the axis of the tube AP, so that the radius vector r makes the angle ϵ with the tube, we have for the normal reaction, from page 123,



$$N = 2mv\omega - mr\omega^2 \sin\epsilon$$
$$= 2mv\omega - ma\omega^2, \quad . \quad . \quad . \quad . \quad (10)$$

which becomes zero when $v = \frac{a\omega}{2}$, and is positive or negative so long as v is greater or less than $\frac{a\omega}{2}$.

For the acceleration along the tube we have

$$\frac{d^2x}{dt^2} = r\omega^2 \cos\epsilon = \omega^2 x,$$

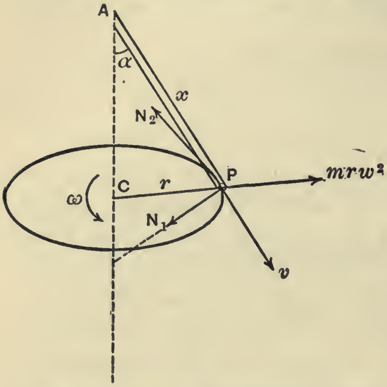
which is precisely the same equation as (1). Hence equations (4), (5), and (6) hold good still, and equation (6) becomes, since $x^2 = r^2 - a^2$,

$$v^2 - v_1^2 = \omega^2r^2 - \omega^2r_1^2,$$

or the difference of the squares of the velocities in the tube equals the difference of the squares of the velocities of rotation.

(8) *Find the motion of a particle in a smooth straight tube which revolves uniformly about a vertical axis which makes an angle with the tube.*

[Ans. Let the tube make the angle α with the axis AC, and r be the radius of rotation at any instant, for which the length of the tube $AP = x$.



The acceleration along AP is

$$\frac{d^2x}{dt^2} = r\omega^2 \sin\alpha,$$

or, since $r = x\sin\alpha$,

$$\frac{d^2x}{dt^2} = (\omega\sin\alpha)^2 x.$$

Comparing with the preceding example, we see that equations (4), (5), (6) hold if we replace ω in these equations by $\omega\sin\alpha$.

We have also the normal reaction $N_1 = mr\omega^2\cos\alpha$, and from page 123 the normal reaction $N_2 = 2mv\omega\sin\alpha$.

(9) *Let the tube rotate uniformly in a vertical plane about a horizontal axis.*

[Ans. We have in this case

$$\frac{d^2x}{dt^2} = \omega^2 x - g\cos\omega t, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

if we conceive the tube to be vertical when $t = 0$. The general integral of this equation is

$$x = Ae^{+\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2}\cos\omega t. \; . \; . \; . \; . \; . \; . \quad (2)$$

Differentiating (2), we have

$$v = \frac{dx}{dt} = A\omega e^{+\omega t} - B\omega e^{-\omega t} - \frac{g}{2\omega}\sin\omega t, \quad . \; . \; . \; . \quad (3)$$

where e is the base of the Naperian system of logarithms and A and B are constants of integration. To determine these constants let $x = x_1$ and $v = v_1$ when $t = 0$. We have then

$$x_1 = A + B + \frac{g}{2\omega^2}, \quad v_1 = A\omega - B\omega.$$

Hence

$$A = \frac{1}{2}\left(x_1 + \frac{v_1}{\omega} - \frac{g}{2\omega^2}\right), \quad B = \frac{1}{2}\left(x_1 - \frac{v_1}{\omega} - \frac{g}{2\omega^2}\right).$$

Substituting in (2) and (3), we have

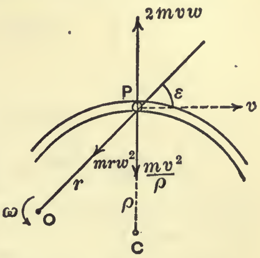
$$x = \frac{1}{2}\left(x_1 + \frac{v_1}{\omega} - \frac{g}{2\omega^2}\right)e^{+\omega t} + \frac{1}{2}\left(x_1 - \frac{v_1}{\omega} - \frac{g}{2\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2}\cos\omega t; \quad (4)$$

$$v = \frac{1}{2}\left(x_1 w + v_1 - \frac{g}{2\omega}\right)e^{+\omega t} - \frac{1}{2}\left(x_1\omega - v_1 - \frac{g}{2\omega}\right)e^{-\omega t} - \frac{g}{2\omega}\sin\omega t. \; . \quad (5)$$

From page 123 we have for the normal reaction of the tube

$$N = 2mv\omega - mg \sin \omega t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(10) *Let the tube be a plane curve rotating uniformly about an axis perpendicular to the plane.*



[Ans. From page 123 we have for the normal reaction due to rotation $2mv\omega$ acting away from the centre of curvature C.

The normal reaction due to the velocity v is $\frac{mv^2}{\rho}$ acting towards the centre of curvature C. The normal reaction due to the deflecting force along PO is $mr\omega^2 \sin \epsilon$ acting towards C. We have then for the normal reaction

$$N = -\frac{mv^2}{\rho} - mr\omega^2 \sin \epsilon + 2mv\omega. \quad . \quad (1)$$

From (1) we see that the normal reaction will be zero when

$$v = \rho\omega \pm \sqrt{\rho^2\omega^2 - \rho r\omega^2 \sin \epsilon}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, for any position of the tube there are in general two velocities for which the normal reaction will be zero.

The tangential acceleration is

$$\frac{dv}{dt} = r\omega^2 \cos \epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we multiply both sides of (3) by $2ds$, we have, since $ds \cos \epsilon = dr$

$$2v\, dv = \omega^2 \,.\, 2r\, dr.$$

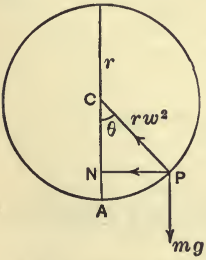
Integrating, and letting $v = v_1$ when $r = r_1$, we have

$$v^2 - v_1^2 = \omega^2(r^2 - r_1^2)$$

or the difference of the squares of the velocities in the tube equals the difference of the squares of the velocities of rotation.

If the tube is a circle, $r = r_1$ and the speed v is constant.

(11) *Let the tube be a circle turning uniformly about a vertical diameter.*



[Ans. The acceleration towards N is

$$NP.\ \omega^2 = r \sin \theta \,.\, \omega^2.$$

The acceleration towards C is then $\frac{NP.\omega^2}{\sin \theta} = r\omega^2$. The vertical component of this is $r\omega^2 \cos \theta$. The vertical acceleration is then $r\omega^2 \cos \theta - g$.

The tangential component is then

$$ft = r\frac{d^2\theta}{dt^2} = (r\omega^2 \cos \theta - g) \sin \theta. \quad . \quad . \quad . \quad (1)$$

Integrating (1), we have

$$\left(\frac{d\theta}{dt}\right)^2 = \omega^2 \sin^2 \theta + \frac{2g}{r} \cos \theta + \text{Const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the particle is projected from the lowest point with the angular velocity ω_1, we have from (2)

$$\omega_1^2 = \frac{2g}{r} + \text{Const.}, \quad \text{or} \quad \text{Const} = \omega_1^2 - \frac{2g}{r}.$$

Hence

$$\left(\frac{d\theta}{dt}\right)^2 = \omega^2\left[1 - \cos^2\theta - \frac{2g}{r\omega^2}(1 - \cos\theta)\right] + \omega_1^2. \quad . \quad . \quad . \quad (3)$$

This will be zero when θ has a value determined by the equation

$$\cos^2\theta - \frac{2g}{r\omega^2}\cos\theta = 1 - \frac{2g}{r\omega^2} + \frac{\omega_1^2}{\omega^2},$$

or

$$\cos\theta = \frac{g}{r\omega^2} \pm \sqrt{\left(1 - \frac{g}{r\omega^2}\right)^2 + \frac{\omega_1^2}{\omega^2}}.$$

So long then as $\frac{\omega_1^2}{\omega^2} > \frac{4g}{r\omega^2}$, or $\omega_1^2 > \frac{4g}{r}$, both values of $\cos\theta$ are numerically greater than 1, and the motion is therefore one of continuous revolution.

If $\omega_1^2 = \frac{4g}{r}$, we have $\frac{d\theta}{dt} = 0$ for $\cos\theta = -1$, and therefore the particle comes to rest at the highest point. In this case the square of the initial velocity is $v_1^2 = r^2\omega_1^2 = 4rg$, or the velocity is that due to the diameter. Hence if a particle is projected from the lowest point with a velocity due to the diameter it will come to rest at the highest point whether the circle is fixed or revolving—a simple instance of conservation of energy (page 87).

If $\omega_1^2 < \frac{4g}{r}$, there is but one possible value of $\cos\theta$, and therefore the particle will oscillate about the lowest point.

The position of equilibrium of the particle is found by putting $\frac{d^2\theta}{dt} = 0$. If we denote the corresponding value of *ACP* by θ', we have

$$\cos\theta' = \frac{g}{r\omega^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To find the time of a small oscillation about this position let ψ be the angle of displacement; then, since $\theta = \theta' + \psi$, and ψ is very small, we have from (1)

$$\begin{aligned}\frac{d^2\psi}{dt^2} &= \omega^2\sin(\theta' + \psi)\left[\cos(\theta' + \psi) - \frac{g}{r\omega^2}\right]\\ &= -\,\omega^2\sin^2\theta'\,.\,\psi, \text{ nearly, by (4),}\\ &= -\left(\omega^2 - \frac{g^2}{r^2\omega^2}\right)\psi.\end{aligned}$$

Multiplying both sides by r, we have the tangential acceleration

$$f_t = -\left(\omega^2 - \frac{g^2}{r^2\omega^2}\right)s.$$

The motion is therefore harmonic, and from page 106, Vol. I, *Kinematics*, the time of oscillation is

$$t = \frac{2\pi r\omega}{\sqrt{r^2\omega^4 - g^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

That there may be a position of equilibrium other than the highest or lowest point, we must have by (4)

$$\omega > \sqrt{\frac{g}{r}}.$$

We see then from (5) that a small oscillation is always possible when there is a position of equilibrium other than the highest or lowest point.

CHAPTER X.

KINETICS OF A SYSTEM. TRANSLATION.

APPLICATION OF LAW OF ENERGY. EXTERNAL AND INTERNAL FORCES. CONSERVATION OF CENTRE OF MASS. CONSERVATION OF MOMENTUM. CONSERVATION OF MOMENTS. CONSERVATION OF AREAS. IMPACT. DIRECT CENTRAL IMPACT. INELASTIC IMPACT. ELASTIC IMPACT. EARTH CONSOLIDATION. PILE-DRIVING. OBLIQUE CENTRAL IMPACT. FRICTION OF IMPACT. STRENGTH OF IMPACT. IMPACT OF BEAMS.

Application of Law of Energy.—We have seen (page 87) that the gain or loss of energy of a particle is equal to the work done by or against the non-conservative forces acting on that particle. Hence for a system of particles the gain or loss of energy of the system must be equal to the algebraic sum of the works done by or against the non-conservative forces acting upon all the particles of the system. The law of energy then applies to systems of particles.

External and Internal Forces.—The forces acting on a system of particles may be divided into two classes, those acting between the particles of a system and external bodies, called external forces, and those acting between the particles of the systems themselves, called internal forces. The internal forces may be mutual attractions, explosive forces, reactions exerted during collision, or the stresses or tensions in connecting strings.

The internal forces between any two particles of a system must always be equal in magnitude and opposite in direction.

Conservation of Centre of Mass.—The motion of the centre of mass of a system is the same as if all the forces were applied without change in magnitude or direction to a particle of mass equal to the mass of the system placed at the centre of mass (page 75, Vol. II, *Statics*).

But since the internal forces between any two particles of a system are equal and opposite, they can have no effect upon the motion of the centre of mass.

The motion of the centre of mass of any system is unaffected by internal forces between the particles of that system.

This is called the principle of "conservation of the centre of mass."

Conservation of Momentum.—Let the particles m_1, m_2, m_3, etc., of a system have velocities v_1, v_2, v_3, etc., in any given direction. Then if M is the combined mass of the system and V the velocity of the centre of mass in that direction, we must have by the preceding principle the *momentum* (page 32) MV of the system equal to the algebraic sum of the momentum of every particle, or

$$MV = m_1v_1 + m_2v_2 + m_3v_3 + \ldots = \Sigma mv.$$

Now, since the motion of the centre of mass is unaffected by internal forces, it follows that

The momentum of any system is unaffected by internal forces between the particles of that system, and is always equal to the algebraic sum of the momentum of the particles.

This is called the principle of "conservation of momentum."

Conservation of Moments.—The force acting upon any particle of a system is the resultant of the external and internal forces acting on that particle. If we take any point as a point of moment, the moment of this resultant is equal to the algebraic sum of its components. But since the internal forces between any two particles of a system are equal in magnitude and opposite in direction, the algebraic sum of the moments of all the internal forces is zero.

Hence, *the algebraic sum of the moments of the forces acting upon all the particles of a system is not affected by the internal forces between the particles of that system, and is always equal to the algebraic sum of the moments of the external forces themselves.*

"This is called the principle of the "conservation of moments."

Conservation of Areas.—Let f be the acceleration of any particle of a system of mass m due to the external force acting upon it, and v the change of velocity in the direction of f in the indefinitely small time t. Then the external force is mf or $\frac{mv}{t}$. Let p be the lever-arm of the force with reference to any point of moments. Then the moment of the external force is mfp or $\frac{mvp}{t}$. This moment, as we have seen, is not affected by the internal forces of the system.

But the moment vp of the velocity is equal to twice the areal velocity of the radius vector, and the moment fp of the acceleration is equal to twice the areal acceleration of the radius vector (page 65, Vol. I, *Kinematics*).

Hence the principle of conservation of moments may be stated as follows:

The algebraic sum of the products of the masses of the particles of a system by the areal velocity or areal acceleration of each radius vector is unaffected by the internal forces.

It follows that *the areal velocity or the areal acceleration of the radius vector of any particle of a system is not affected by the internal forces of the system.*

This is called the principle of "conservation of areas."

EXAMPLES.

(1) *Two particles of masses m_1 and m_2 at a distance s_1 are initially at rest on a smooth horizontal plane, and attract each other with uniform force. After a time t the greater mass m_2 has a velocity v_2. Find the velocity v_1 of the mass m_1, the internal force, the distance s apart at the end of the time t, and the position of the centre of mass.*

Ans. Let v be the velocity of the centre of mass. Then, since there are no external forces and v_1 and v_2 are opposite in direction, we have by the conservation of momentum

$$(m + m_1)v = m_1v_1 - m_2v_2.$$

But the centre of mass is originally at rest, and, since there are no external forces, by the conservation of the centre of mass it must remain at rest. Hence $v = 0$, and

$$m_1v_1 - m_2v_2 = 0, \quad \text{or} \quad v_1 = \frac{m_2v_2}{m_1}.$$

Since the internal force is uniform, the distance passed over by m_1 is $\frac{v_1}{2}t$, and by m_2, $\frac{v_2}{2}t$. The distance apart is then

$$s = s_1 - \frac{v_1 + v_2}{2}t.$$

The internal force is $\frac{m_1v_1}{t}$ or $\frac{m_2v_2}{t}$ poundals. Since these forces are equal and opposite, we have

$$\frac{m_1v_1}{t} - \frac{m_2v_2}{t} = 0, \quad \text{or, as before,} \quad m_1v_1 - m_2v_2 = 0.$$

The distance of the centre of mass from m_1 at the start is $\frac{m_2}{m_1 + m_2}s_1$, and from m_2, $\frac{m_1}{m_1 + m_2}s_1$. At the end, $\frac{m_2}{m_1 + m_2}s$ and $\frac{m_1}{m_1 + m_2}s$.

If $m_1 = 50$ lbs., $m_2 = 100$ lbs., $v_2 = 10$ ft. per sec., $t = 1/20$ sec., $s_1 = 3$ ft., we have

$v_1 = 20$ ft. per sec., $s = 2.25$, force $= 50$ poundals, distance of centre of mass from m_1 and m_2 at start 2 ft. and 1 ft., and at end 1.5 ft. and 0.75 ft.

(2) *In the preceding example suppose the particles have an initial angular velocity about the centre of mass of* ω_1 *radians per sec. Find the final angular velocity* ω.

Ans. Let r_1 be the distance of m_1 from the centre of mass at the start, and r its distance at the end. Then the areal velocity of the radius vector at the start is $r_1^2\omega_1$, and at the end $r^2\omega$. There are no external forces, and by the conservation of areas the areal velocity of the radius vector is not affected by internal forces. We have then

$$r^2\omega = r_1^2\omega_1, \quad \text{or} \quad \omega = \frac{r_1^2}{r^2}\omega.$$

From the preceding example, $r_1 = \frac{m_2}{m_1 + m_2}s_1$ and $r = \frac{m_2}{m_1 + m_2}s$. Hence

$$\omega = \frac{s_1^2}{s^2}\omega_1.$$

Taking the numerical values of the preceding example,

$$\omega = \frac{3^2}{2.25^2}\omega_1 = \frac{9}{5.0625}\omega_1 = 1\tfrac{7}{9}\omega_1.$$

We see then that *the angular velocity increases as the particles approach the centre of mass.*

(3) *What effect has the bursting of a bomb upon the motion of its centre of mass?*

Ans. None whatever. By the law of conservation of the centre of mass, the motion of the centre of mass of the system, neglecting all resistances of the air, etc., and all external forces, is not affected.

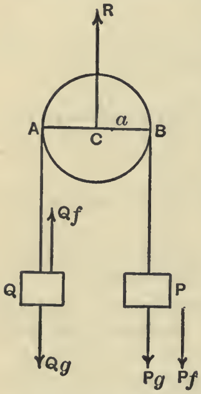
(4) *A projectile of mass* m_1 *is thrown with a velocity* v_1 *from a cannon of mass* m_2. *Find the velocity of recoil of the cannon.*

Ans. The motion of the centre of mass of the system is not affected by the explosion. We have then, since the velocities are in different directions,

$$m_1v_1 - m_2v_2 = 0, \quad \text{or} \quad v_2 = \frac{m_1}{m_2}v_1.$$

See also example 20, page 64.

(5) *Two masses P and Q hang over a smooth pulley by means of a perfectly flexible inextensible string without mass. Disregarding the mass of the pulley, find the motion. (The student should compare with the solutions of pages* 8 and 53.)



Ans. Let a be the radius of the pulley and P the larger mass. By the conservation of moments the algebraic sum of the moments of the forces acting upon all the particles is unaffected by internal forces and equal to the algebraic sum of the moments of the external forces.

The external forces are Pg and Qg acting down and the reaction R acting up at the centre of the pulley. Let f be the acceleration of P and Q. Then the forces acting on the particles are Pf acting down and Qf acting up. If then we take C as the centre of moments we have

$$-Pfa - Qfa = -Pga + Qga,$$

or

$$f = \frac{(P-Q)g}{P+Q}.$$

The tension of the string on the left is $Q(g-f)$, and of the string on the right $P(g-f)$. (See example 1, page 53.)

The reaction R is then

$$R = Q(g+f) + P(g-f).$$

If then we take moments about B, we have

$$-Qf \times 2a = Qg \times 2a - Ra.$$

If we take moments about A, we have

$$-Pf \times 2a = -Pg \times 2a + Ra.$$

If we substitute the value of R, we have in both cases, just as before,

$$f = \frac{(P-Q)g}{P+Q}.$$

(6) *In the preceding example take friction of the axle into account.* (*See Example* 5, *page* 77.)

Impact.—When two moving bodies come in collision the straight line normal to the surfaces at the point of contact is the line of impact. If the centre of mass of the two bodies is upon this line, the impact is called **central impact**; if not, we have **eccentric impact.**

When we consider the direction of motion, we can distinguish **direct impact** when the line of impact coincides with the direction of motion, and **oblique impact** when the line of impact does not coincide with the direction of motion.

Thus in Fig. 1 if the two bodies move in the directions u_1 and u_2 we have *oblique central* impact, and in Fig. 2 we have *oblique eccentric* impact. If in Fig. 1 the directions of motion u_1 and u_2 coincided with C_1C_2 we should have *direct central* impact. If in Fig. 2 the direction of motion u_1 coincided with C_1N, and u_2 were parallel, we should have *direct eccentric* impact.

FIG. 1.

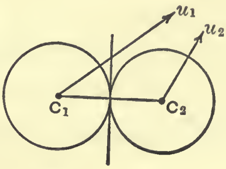
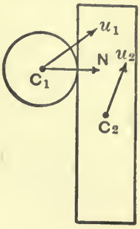
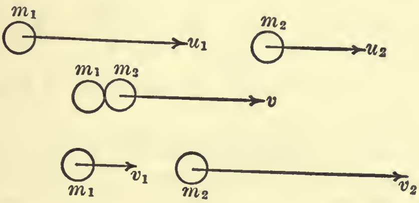


FIG. 2.



Direct Central Impact—General Equation.—We can evidently consider the bodies in direct central impact as particles. Let m_1 and u_1 be the mass and initial velocity of one particle before impact and m_2 and u_2 the mass and initial velocity of the other before impact. Let u_1 be greater than u_2 and in the same direction. Let the direction of u_1 be positive, the opposite direction negative.

When the particles meet there is a short interval of compression, at the end of which both masses have the common velocity v. If the particles are inelastic they remain in contact with this velocity. If they are elastic there is another short interval of expansion, at the end of which m_1 has the final velocity v_1 less than u_1, and m_2 the final velocity v_2 greater than u_2. All velocities in any given direction, as the direction of u_1, are to be taken as positive and in the opposite direction negative.



Now by the principle of conservation of centre of mass, since there are no external forces, the motion of the centre of mass is unaffected by impact and is constant both before, during and after impact. Also by the principle of conservation of momentum the momentum of the system is always equal to the algebraic sum of the momentum of the particles.

We have then before impact, if v is the velocity of the centre of mass, which must be the same as the common velocity at the end of compression,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and after impact

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we have for the common velocity of the bodies at the end of the period of compression, or the uniform velocity of the centre of mass,

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence

$$\left.\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ \text{or} \qquad m_1u_1 - m_1v_1 &= m_2v_2 - m_2u_2. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(I)}$$

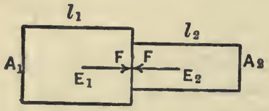
That is,

the momentum before equals the momentum after impact, or

the momentum lost by one body equals the momentum gained by the other.

In equations (I) velocities opposite in direction to u_1 are to be taken as negative.

Compression and Compressive Stress.—Let the cross-section of the bodies be A_1 and A_2, their lengths l_1 and l_2, and coefficients of elasticity E_1 and E_2. Then if the limit of elasticity is not exceeded we have by the law of elasticity (page 281, Vol. II, *Statics*), for the compressive strains,



$$\lambda_1 = \frac{Fl_1}{A_1E_1}, \quad \lambda_2 = \frac{Fl_2}{A_2E},$$

where F is the compressive stress between the two bodies.

For the sake of simplicity we can put

$$\frac{F}{\lambda_1} = \frac{A_1E_1}{l_1} = H_1, \quad \frac{F}{\lambda_2} = \frac{A_2E_2}{l_2} = H_2. \quad . \; . \; . \; . \; . \; . \quad (4)$$

We can call the quantity $\frac{AE}{l}$ the **hardness** of a body.

The "hardness" of a body, then, is measured by the ratio of the stress in pounds to the resulting strain in inches or feet, provided the limit of elasticity is not exceeded. It is given then in pounds per inch or pounds per foot. We have then, in general,

$$\lambda_1 = \frac{F}{H_1}, \qquad \lambda_2 = \frac{F}{H_2}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

where H_1 and H_2 are the hardness of the bodies as given by equations (4), F the compressive stress between them, and λ_1, λ_2 the respective compressive strains.

We have then for the total compressive strain

$$\lambda_1 + \lambda_2 = \frac{H_1 + H_2}{H_1H_2}F, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and for the compressive stress

$$F = \frac{H_1H_2}{H_1 + H_2}(\lambda_1 + \lambda_2). \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

Since the work of compression is one half the product of stress and strain (page 281, Vol. II, *Statics*), we have for the loss of energy during compression.

$$\frac{1}{2}F(\lambda_1 + \lambda_2) = \frac{H_1H_2}{2(H_1 + H_2)}(\lambda_1 + \lambda_2)^2. \quad . \; . \; . \; . \; . \; . \quad (8)$$

Now E_1 and E_2 are given in our Table (page 290, Vol. II, *Statics*) in pounds *per square inch*. If then we always take A_1 and A_2 in square inches, A_1E_1 and A_2E_2 will always give pounds. If we then always take l_1 and l_2 in feet, we shall have H_1 and H_2 in terms of *pounds per foot.* If then we take λ_1 and λ_2 in feet, equation (8) gives the loss of energy in *foot-pounds.* To reduce to foot-poundals, we must then multiply by g in ft.-per-sec. per sec. We have then for the loss of energy during compression, in *foot-poundals,*

$$\frac{1}{2}Fg(\lambda_1 + \lambda_2) = \frac{H_1H_2g}{2(H_1 + H_2)}(\lambda_1 + \lambda_2)^2. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

But we also have for the loss of energy during compression, *in foot-poundals,*

$$\frac{1}{2}Fg(\lambda_1 + \lambda_2) = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2)v^2.$$

Inserting the value of v from (3), this becomes

$$\frac{1}{2}Fg(\lambda_1 + \lambda_2) = \frac{m_1m_2(u_1 - u_2)^2}{2(m_1 + m_2)}.$$

Equating this to (9), we obtain for the total compressive strain

$$\lambda_1 + \lambda_2 = (u_1 - u_2)\sqrt{\frac{m_1m_2}{(m_1 + m_2)g} \cdot \frac{H_1 + H_2}{H_1H_2}}; \quad . \quad . \quad . \quad (\mathrm{II})$$

and hence from (7)

$$F = (u_1 - u_2)\sqrt{\frac{m_1m_2}{(m_1 + m_2)g} \cdot \frac{H_1H_2}{H_1 + H_2}}; \quad . \quad . \quad . \quad (\mathrm{III})$$

also from (5)

$$\left.\begin{aligned} \lambda_1 &= (u_1 - u_2)\sqrt{\frac{m_1m_2}{(m_1 + m_2)g} \cdot \frac{H_2}{H_1(H_1 + H_2)}}; \\ \lambda_2 &= (u_1 - u_2)\sqrt{\frac{m_1m_2}{(m_1 + m_2)g} \cdot \frac{H_1}{H_2(H_1 + H_2)}}. \end{aligned}\right\} \quad (\mathrm{IV})$$

In all these three equations λ_1 and λ_2 are in feet, F in pounds, u_1 and u_2 in ft. per sec., g in ft.-per-sec. per sec., m_1 and m_2 in pounds, and H_1 and H_2 in pounds per foot. That is, E_1 and E_2 are taken in pounds per square inch from our Table (page 290, Vol. II, *Statics*), A_1 and A_2 are taken in square inches, and l_1 and l_2 in feet. If u_2 has a direction opposite to u_1 it is to be taken as negative.

Modulus of Elasticity.—Let F be the compressive stress upon a body and λ the corresponding strain. When F is removed the body expands. Let then F' be the stress of restitution and λ' its corresponding strain. The ratio $\frac{F'}{F}$ of the stress of restitution to the stress of compression is found by experiment to be a constant for any given material, as long as the limit of elasticity is not exceeded. This ratio we denote by e and call the **modulus of elasticity.** But if the limit of elasticity is not exceeded, the stress and strain are proportional. We have then

$$\frac{F'}{F} = \frac{\lambda'}{\lambda}e.$$

If the body is perfectly elastic, $F' = F$ and $\lambda' = \lambda$, or the body perfectly recovers its original dimensions. We have then $e = 1$. If the body is non-elastic, $F' = 0$ and $\lambda' = 0$, and $e = 0$. For imperfectly elastic bodies e is less than 1 and λ' less than λ, and the body does not completely recover its original dimensions.

Imperfectly Elastic Impact.—When two bodies come into collision let F be the stress during compression and λ_1, λ_2, the corresponding strains. Let the respective stresses of restitution during the period of expansion be F_1' and F_2', and λ_1', λ_2' the respective strains. Let e_1 and e_2 be the respective moduli of elasticity. Then we have

$$\frac{F_1'}{F} = \frac{\lambda_1'}{\lambda_1} = e_1 \quad \text{and} \quad \frac{F_2'}{F} = \frac{\lambda_2'}{\lambda_2} = e_2.$$

Hence

$$F_1'\lambda_1' = e_1^2 F\lambda_1 \quad \text{and} \quad F_2'\lambda_2' = e_2^2 F\lambda_2. \quad . \quad . \quad . \quad . \quad (10)$$

The loss of energy during the entire period of impact is then, since work equals one half the product of stress and strain (page 281, Vol. II, *Statics*), from (10),

$$\frac{1}{2}F(\lambda_1 + \lambda_2) - \frac{1}{2}F_1'\lambda_1' - \frac{1}{2}F_2'\lambda_2' = \frac{1}{2}F[(1 - e_1^2)\lambda_1 + (1 - e_2^2)\lambda_2];$$

or, since from (5)

$$\lambda_1 = \frac{F}{H_1}, \quad \lambda_2 = \frac{F}{H_2},$$

we have for the loss of energy L during the entire period of impact

$$L = \frac{1}{2}F^2\left[\frac{1 - e_1^2}{H_1} + \frac{1 - e_2^2}{H_2}\right].$$

If we take F in pounds and H_1, H_2 in pounds per foot, this is the loss of energy in foot-pounds. For the loss of energy in foot-poundal, then, we have

$$L = \frac{1}{2}F^2 g\left[\frac{1 - e_1^2}{H_1} + \frac{1 - e_2^2}{H_2}\right];$$

or, if we insert the value of F from (III),

$$L = \frac{(u_1 - u_2)^2}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{H_1 H_2}{H_1 + H_2}\left[\frac{1 - e_1^2}{H_1} + \frac{1 - e_2^2}{H_2}\right]. \quad (V)$$

But this loss of energy is also given in foot-poundals by

$$L = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_2 v_2^2.$$

Equating then these two expressions, we have

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 =$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{(u_1 - u_2)^2}{2} \cdot \frac{m_1m_2}{m_1 + m_2} \cdot \frac{(1 - e_1^2)H_2 + (1 - e_2^2)H_1}{H_1 + H_2}.$$

If we eliminate v_2 and v_1 by (I), viz.,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2,$$

we obtain

$$\left.\begin{aligned} u_1 - v_1 &= (u_1 - u_2)\frac{m_2}{m_1 + m_2}\left[1 + \sqrt{\frac{e_2^2H_1 + e_1^2H_2}{H_1 + H_2}}\right]; \\ v_2 - u_2 &= (u_1 - u_2)\frac{m_1}{m_1 + m_2}\left[1 + \sqrt{\frac{e_2^2H_1 + e_1^2H_2}{H_1 + H_2}}\right]. \end{aligned}\right\} \quad \text{(VI)}$$

In equations (VI) we take m_1 and m_2 in pounds, H_1 and H_2 in pounds per foot and u_1, u_2, v_1, v_2 in feet per second. Velocities in the direction of u_1 are positive, in the opposite direction negative. If the bodies are non-elastic $e_1 = 0$, $e_2 = 0$. If the bodies are perfectly elastic $e_1 = e_2 = 1$. If the two bodies are *of the same material* $e_1 = e_2 = e$ and we have

$$\left.\begin{aligned} v_1 &= u_1 - (u_1 - u_2)\frac{(1 + e)m_2}{m_1 + m_2}; \\ v_2 &= u_2 + (u_1 - u_2)\frac{(1 + e)m_1}{m_1 + m_2}. \end{aligned}\right\} \quad . \; . \; . \; . \; \text{(VII)}$$

Experimental Determination of Modulus of Elasticity.—Let the mass m_2 be rigidly fixed so that $u_2 = 0$, $v_2 = 0$. Then from the second of equations (VI) $m_2 = \infty$ and from the first of equations (VI) we have

$$e = -\frac{v_1}{u_1}.$$

If then we cause a sphere of mass m_1 to fall from a height h upon a rigidly supported flat mass m_2 of the same material, and if it bounds back to a height h', we have $u_1 = \sqrt{2gh}$ and $v_1 = -\sqrt{2gh'}$. Hence

$$e = \frac{\sqrt{h'}}{\sqrt{h}} = \sqrt{\frac{h'}{h}}.$$

We can thus determine the modulus of elasticity for various materials.

We have thus the average values:

cast iron, $e = 1$, nearly;
glass, $e = 15/16$;
ivory, $e = 8/9$;
cork, steel, $e = 5/9$;
clay, wood, $e = 0$, nearly.

Non-elastic Impact.—The preceding formulas (I) to (VII) are general and include all special cases.

For non-elastic impact equations (I), page 146, and (II) to (IV), page 147, hold good without change; and since $e_1 = 0$, $e_2 = 0$, equations (VI), page 149, become

$$v_1 = v_2 = v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the bodies move together after impact with the common velocity v. The loss of energy we have already found (page 147) equal to

$$L = \frac{m_1m_2(u_1 - u_2)^2}{2g(m_1 + m_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in *foot-pounds.*

We call $\frac{m_1m_2}{m_1 + m_2}$ the *harmonic mean* between m_1 and m_2.

Hence, *the loss of energy during the impact of two inelastic bodies is equal to the product of the harmonic mean of the two masses and the height due to the difference of their velocities.*

If the mass m_2 is at rest, the loss of energy becomes in foot-pounds

$$L = \frac{m_1m_2}{m_1 + m_2} \cdot \frac{u_1^2}{2g},$$

and if the moving mass m_1 is very great compared to the mass m_2 at rest this becomes

$$m_2\frac{u_1^2}{2g}.$$

We have from equation (1)

$$m_1(u_1 - v) = m_2(v - u_2),$$

or the momentum lost equals the momentum gained; and also from equation (1), for the loss and gain of velocity,

$$u_1 - v = \frac{m_2(u_1 - u_2)}{m_1 + m_2}, \quad v - u_2 = \frac{m_1(u_1 - u_2)}{m_1 + m_2}. \quad . \quad . \quad (3)$$

The energy lost, then, is evidently given in foot-pounds by

$$\frac{1}{2g}m_1(u_1 - v)^2 + \frac{1}{2g}m_2(u_2 - v)^2.$$

Special Cases.—If the mass m_2 is at rest, we have $u_2 = 0$ and

$$v = \frac{m_1u_1}{m_1 + m_2}, \quad L = \frac{m_1m_2}{m_1 + m_2} \cdot \frac{u_1^2}{2g}.$$

If the bodies move towards each other, u_2 is negative and

$$v = \frac{m_1u_1 - m_2u_2}{m_1 + m_2}, \quad L = \frac{m_1m_2}{m_1 + m_2} \cdot \frac{(u_1 + u_2)^2}{2g}.$$

In this case, if the momenta of the bodies are equal, or $m_1u_1 = m_2u_2$, $v = 0$, or the bodies come to rest. If, on the contrary, the masses are equal, we have

$$v = \frac{u_1 - u_2}{2}, \quad L = \frac{m}{2} \cdot \frac{(u_1 + u_2)^2}{2g}.$$

If the bodies move in the same direction and the mass of the one in advance, m_2, is infinitely great, we have

$$v = u_2, \qquad L = \frac{m_1(u_1 - u_2)^2}{2g},$$

or the velocity of the infinitely great body is not changed by the impact. If the infinitely great mass is at rest, or $u_2 = 0$, we have

$$v = 0, \qquad L = \frac{m_1u_1^2}{2g},$$

and the infinitely great body remains at rest, while the impinging body loses its velocity entirely.

Perfectly Elastic Impact.—Equations (I), page 146, and (II) to (IV), page 147, hold good without change; and since for perfectly elastic bodies $e_1 = 1$, $e_2 = 1$, equations (VI), page 149, become

$$\left.\begin{aligned} v_1 &= u_1 - \frac{2m_2(u_1 - u_2)}{m_1 + m_2}; \\ v_2 &= u_2 + \frac{2m_1(u_1 - u_2)}{m_1 + m_2}. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (1)$$

The loss of energy, we see from equation (V), page 148, is zero. That is, *there is no loss of energy in perfectly elastic impact.*

We have then

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}m_1v_1^2 - \frac{1}{2}m_1v_2^2 = 0,$$

or

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2).$$

But since from (I), page 146, we have

$$m_1(u_1 - v_1) = m_2(v_2 - u_2),$$

we have, by eliminating m_1 and m_2,

$$\frac{u_1^2 - v_1^2}{u_1 - v_1} = \frac{v_2^2 - u_2^2}{v_2 - u_2}, \quad \text{or} \quad u_1 + v_1 = v_2 + u_2,$$

or

$$u_1 - u_2 = v_2 - v_1.$$

That is, *the velocity of approach equals the velocity of separation.*

The loss and gain of velocity are then

$$u_1 - v_1 = \frac{2m_2(u_1 - u_2)}{m_1 + m_2}, \quad v_2 - u_2 = \frac{2m_1(u_1 - u_2)}{m_1 + m_2}, \quad . \ . \quad (3)$$

or *twice as much* as for non-elastic impact (page 150).

Special Cases.—If the mass m_2 is at rest, we have $u_2 = 0$ and

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1, \quad v_2 = \frac{2m_1}{m_1 + m_2} u_1.$$

If the bodies move towards each other, u_2 is negative and

$$v_1 = u_1 - \frac{2m_2(u_1 + u_2)}{m_1 + m_2}, \quad v_2 = -u_2 + \frac{2m_1(u_1 + u_2)}{m_1 + m_2}.$$

In this case, if the momenta of the bodies are equal, or $m_1u_1 = m_2u_2$, we have

$$v_1 = -u_1, \quad v_2 = +u_2;$$

that is, the bodies after impact move in opposite directions with the same velocities they originally had. If, on the contrary, the masses are equal, we have

$$v_1 = -u_2, \quad v_2 = u_1;$$

that is, each body returns with the same velocity that the other body had before impact.

If the bodies move in the same direction, and the mass m_2 of the one in advance is infinitely great, we have

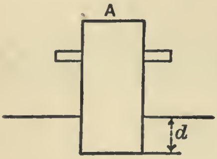
$$v_1 = u_2,$$

or the velocity of the infinitely great body is not changed by the impact. If the infinitely great body is at rest, or $u_2 = 0$, we have

$$v_1 = -u_1, \quad v_2 = 0;$$

that is, the velocity of the impinging body is transformed into an equal opposite one.

Earth Consolidation.—When a maul strikes a mass of soft earth it compresses it with a certain force F. Let d be the depth of penetration, and m the mass of the maul and h the height from which it is let fall. Then the energy of the maul before it is dropped is mh. Since this energy is expended in compression of the soil, we have



$$Fs = mh, \quad \text{or} \quad F = \frac{mh}{s}.$$

If we divide this force F by the cross-section A of the maul, we have for the unit force of compression

$$p = \frac{F}{A} = \frac{mh}{As}.$$

The resistance F of soils to the penetration of a maul is generally variable and increases with the depth d of penetration. In many cases we may assume it to increase directly with the penetration. In such case we should have

$$\frac{1}{2} Fs = mh, \quad \text{or} \quad F = \frac{2mh}{s};$$

or

$$p = \frac{F}{A} = \frac{2mh}{As},$$

or twice as much as before.

If A is taken in square inches and s and h are taken in feet or inches, p is the number of pounds per square inch, resistance of the soil. Allowing a factor of safety of 10, we could then safely load the compacted soil up to $\frac{1}{10}p$.

Pile-driving.—We see from equation (6), page 146, that where two bodies impinge the compression is given by

$$\lambda_1 + \lambda_2 = \left(\frac{1}{H_1} + \frac{1}{H_2}\right)F, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where F is the compressive stress between the bodies, λ_1 the compression of one, λ_2 the compression of the other, and H_1 and H_2 the *hardness* (page 146), so that

$$H_1 = \frac{A_1E_1}{l_1}, \quad H_2 = \frac{A_2E_2}{l_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A_1 and A_2 are the areas of cross-section of the bodies, l_1 and l_2 their lengths, and E_1, E_2 their coefficients of elasticity. Since E_1 and E_2 are given in our table (page 290, Vol. II, *Statics*) in pounds *per square inch*, if we always take A_1 and A_2 in square inches, A_1E_1 and A_2E_2 will always give pounds. If then we take l_1 and l_2 in feet, H_1 and H_2 will be given in pounds per foot; and if we take F in pounds, equation (1) gives $\lambda_1 + \lambda_2$ in feet.

If now the mass of the impinging body is m_1 and its velocity u_1, its energy in foot-pounds is $\frac{m_1u_1^2}{2g}$.

If the other body is at rest its initial energy is zero.

Since work is equal to one half the product of the stress and strain (page 281, Vol. II, *Statics*), we have for the work expended in compression

$$\frac{1}{2}F(\lambda_1 + \lambda_2) = \frac{1}{2}F^2\left(\frac{1}{H_1} + \frac{1}{H_2}\right).$$

If then a bolt or nail is struck by a hammer of mass m_1, so long as

$$\frac{m_1u_1^2}{2g} = \frac{1}{2}F^2\left(\frac{1}{H_1} + \frac{1}{H_2}\right)$$

all the energy of the hammer is expended in compression and there is no penetration. If, however,

$$\frac{m_1u_1^2}{2g} > \frac{1}{2}F^2\left(\frac{1}{H_1} + \frac{1}{H_2}\right)$$

there will be penetration.

The same holds for the driving of a pile. Let m_1 be the mass of the ram and h the height of fall $= \frac{u_1^2}{2g}$. Then the energy of the ram is m_1h. Let d be the depth of penetration and $\lambda_1 + \lambda_2$ the compression of ram and pile, and F the compressive stress. Then we have

$$Fd + \frac{1}{2}F(\lambda_1 + \lambda_2) = m_1h.$$

Let m_2 be the mass of the pile. Then from equation (II), page 147, we have for $u_2 = 0$ and $\frac{u_1^2}{2g} = h$,

$$\lambda_1 + \lambda_2 = \sqrt{\frac{2m_1m_2h}{m_1 + m_2} \cdot \frac{H_1 + H_2}{H_1H_2}}.$$

Inserting this value of $\lambda_1 + \lambda_2$, we have

$$F = \frac{m_1h}{d + \sqrt{\frac{m_1m_2h}{2(m_1 + m_2)} \cdot \frac{H_1 + H_2}{H_1H_2}}} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

From equation (3) we can find the resistance of the pile by measuring the distance of penetration d.

Since the pile is wood and very long compared to the ram, and the ram is iron, H_1 is very large compared to H_2, and we have approximately

$$\frac{H_1 + H_2}{H_1H_2} = \frac{1 + \frac{H_2}{H_1}}{H_2} = \frac{1}{H_2}, \text{ approximately.}$$

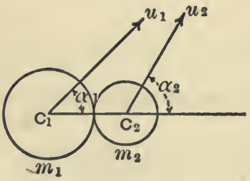
Hence we can write practically

$$F = \frac{m_1h}{d + \sqrt{\frac{m_1m_2h}{2(m_1 + m_2)H_2}}}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

where m_1 and m_2 are to be taken in pounds, h and d in feet or inches, and H_2 in pounds per foot or inch. For wood we can take $E_2 = 1500000$ lbs. per sq. inch (page 290, Vol. II, *Statics*).

If we take a factor of safety of 6 or 10 we can safely load the pile up to $\frac{1}{6}$ or $\frac{1}{10}$ F as given by (3) or (4).

Oblique Central Impact.—If the directions of motion u_1, u_2 of the two bodies make the angles α_1, α_2 with the line of impact C_1C_2, we can resolve each velocity into components $u_1 \cos \alpha_1$ and $u_2 \cos \alpha_2$ along the line of impact and $u_1 \sin \alpha_1$, $u_2 \sin \alpha_2$ at right angles to this line. These latter are unchanged by the impact. As to the former, we have from equations (VII), page 149, if the two bodies are of *the same material* and e is the modulus of elasticity,



$$\left.\begin{aligned} v_1 &= u_1 \cos \alpha_1 - (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)\frac{m_2(1 + e)}{m_1 + m_2}, \\ v_2 &= u_2 \cos \alpha_2 + (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)\frac{m_1(1 + e)}{m_1 + m_2}, \end{aligned}\right\} \quad . \; . \quad (1)$$

where m_1 and m_2 are the masses of the two bodies and v_1, v_2 are the final velocities along the line of impact C_1C_2.

From v_1 and $u_1 \sin \alpha_1$ we have for the velocity w_1 of the first body after impact

$$w_1 = \sqrt{v_1^2 + u_1^2 \sin^2 \alpha_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

making an angle β_1 with C_1C_2 given by

$$\tan \beta_1 = \frac{u_1 \sin \alpha_1}{v_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for the velocity w_2 of the second body after impact

$$w_2 = \sqrt{v_2^2 + u_2^2 \sin^2 \alpha_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

making an angle β_2 with C_1C_2 given by

$$\tan \beta_2 = \frac{u_2 \sin \alpha_2}{v_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the mass m_2 is infinitely great and at rest we have $m_2 = \infty$, $u_2 = 0$, and from (1)

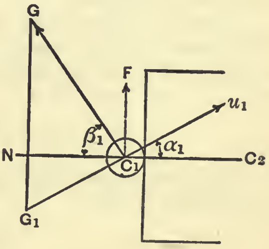
$$\left. \begin{aligned} v_1 &= - e u_1 \cos \alpha_1, \\ v_2 &= 0, \end{aligned} \right\} \quad . \quad . \quad . \quad (6)$$

and from (2) and (4), $w_2 = 0$,

$$w_1 = \sqrt{u_1^2(\sin^2 \alpha_1 + e^2 \cos^2 \alpha_1)}, \quad . \quad . \quad (7)$$

making the angle β_1 with C_1C_2 given by

$$\tan \beta_1 = - \frac{\sin \alpha_1}{e \cos \alpha_1} = - \frac{1}{e} \tan \alpha_1. \quad (8)$$



For inelastic bodies $e = 0$, and for perfectly elastic bodies $e = 1$.

For inelastic bodies, then, from (6), (7) and (8), $v_1 = 0$, $w_1 = u_1 \sin \alpha_1$, $\tan \beta_1 = \infty$, or $\beta = 90°$. That is, the velocity along the line of impact is completely annihilated and that at right angles is unchanged, and the body moves after impact in the direction C_1F at right angles to C_1C_2 with the velocity $u_1 \sin \alpha_1$.

For perfectly elastic bodies $v_1 = - u_1 \cos \alpha_1$, $w_1 = u_1$, $\tan \beta_1 = - \tan \alpha_1$, or $\beta_1 = - \alpha_1$. That is, the velocity along the line of impact is changed into an equal and opposite one, and *the angle of incidence α_1 is equal to the angle of reflection β_1*. The body moves after impact in the direction C_1G so that the angle $NC_1G = \alpha_1$.

For imperfect elasticity we have from (8)

$$e = - \frac{\tan \alpha_1}{\tan \beta_1},$$

or the modulus of elasticity is equal to the ratio of the tangent of the angle of incidence to the tangent of the angle of reflection. We have then for perfect elasticity $NG_1 = NG$, and for imperfect elasticity $\frac{NG_1}{NG} = e$.

Friction of Oblique Central Impact.—The pressure between the colliding bodies gives rise to friction. If P is the pressure due to impact, F the friction and μ the coefficient of friction, then we have

$$F = \mu P.$$

Let the mass of the impinging body be m_1, and the initial and final velocity along the line of impact be u_1 and v_1, and t be the time of impact. Then we have for the *impulse* (page 31)

$$Pt = m_1(u_1 - v_1), \quad \text{or} \quad P = \frac{m_1(u_1 - v_1)}{t}.$$

Hence the friction is

$$F = \frac{\mu m_1(u_1 - v_1)}{t}, \quad \text{or} \quad \frac{Ft}{m_1} = \mu(u_1 - v_1). \quad . \quad . \quad . \quad (1)$$

That is,

the impulse of the friction divided by the mass is equal to μ times the change of velocity along the line of impact, or

the change of velocity due to friction at right angles to the line of impact is equal to μ times the change of velocity along the line of impact.

This change of velocity is always a retardation, since friction is a retarding force.

Thus if a mass m_1 falls vertically with a velocity u_1 upon a horizontal sled of mass m_2 moving with the velocity u_2, and if the velocity u_1 is entirely lost by the collision, we have for the friction

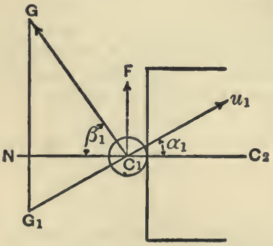
$$F = \frac{\mu m_1 u_1}{t}.$$

But the retarding force during the time t for both masses in contact is also

$$F = \frac{(m_1 + m_2)u_2}{t}.$$

Hence we have

$$u_2 = \mu \frac{m_1}{m_1 + m_2} u_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$



If a body of mass m_1 strikes an immovable mass of the same material with a velocity u_1 at an angle α_1, we have from equation (1), page 154, for the change of velocity along the line of impact, since $u_2 = 0$, $m_2 = \infty$,

$$u_1 \cos \alpha_1 - v_1 = u_1 \cos \alpha_1(1 + e).$$

Hence, the change of velocity due to friction is

$$\mu u_1 \cos \alpha_1(1 + e),$$

and after impact the component $u_1 \sin \alpha_1$ becomes

$$u_1 \sin \alpha_1 - \mu u_1 \cos \alpha_1(1 + e) = [\sin \alpha_1 - \mu \cos \alpha_1(1 + e)]u_1. \quad (3)$$

For perfectly elastic bodies $e = 1$ and (3) becomes

$$(\sin \alpha_1 - 2\mu \cos \alpha_1)\mu_1,$$

and for inelastic bodies

$$(\sin \alpha_1 - \mu \cos \alpha_1)u_1.$$

The friction often causes bodies to turn around their centres of mass, or if before impact a motion of rotation exists, that motion is changed. Let R be the radius of a round body, ω_1 its initial and ω its final angular velocity during the time t of impact. Then the initial and final velocity of any particle at a distance r from the centre of mass will be $r\omega_1$ and $r\omega$. The change of velocity will be $r(\omega - \omega_1)$, the acceleration $\frac{r(\omega - \omega_1)}{t}$, and the particle force, if m is the mass of the particle, is then $\frac{mr(\omega - \omega_1)}{t}$. The moment of this particle force is $\frac{mr^2(\omega - \omega_1)}{t}$, and the sum of the moments of all the particle forces is then

$$\frac{\omega - \omega_1}{t} \Sigma mr^2.$$

But (page 170) Σmr^2 is the moment of inertia I of a body, and hence the sum of the moments of all the particle forces is

$$\frac{\omega - \omega_1}{t} . I.$$

The change of velocity of the body of mass m_1 due to friction, we have just seen, is $\mu u_1 \cos \alpha_1(1 + e)$. Its acceleration is then $\frac{\mu u_1 \cos \alpha_1(1 + e)}{t}$, and the force of friction is then

$$\frac{\mu m_1 u_1 \cos \alpha_1(1 + e)}{t}.$$

The moment of this force is then

$$\frac{\mu R m_1 u_1 \cos \alpha_1(1 + e)}{t}.$$

Now by the principle of conservation of moments (page 142) the sum of the moments of the particle forces is equal to the moment of the friction.

Hence we have

$$\omega - \omega_1 = \frac{\mu R m_1 u_1 \cos \alpha_1(1 + e)}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) gives the change of angular velocity. For the change of linear velocity at the circumference we have

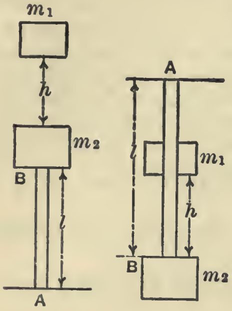
$$R(\omega - \omega_1) = \frac{\mu R^2 m_1 u_1 \cos \alpha_1(1 + e)}{I},$$

or, since $I = m_1\kappa_2$, where κ is the radius of gyration (page 176),

$$R(\omega - \omega_1) = \mu \frac{R^2}{\kappa^2}(1 + e)u_1 \cos \alpha_1 . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For a cylinder (page 177) $\frac{R^2}{\kappa^2} = 2$, and for a sphere (page 178) $\frac{R^2}{\kappa^2} = \frac{5}{2}$.

Strength of Impact.—Let the mass m_1 moving with the velocity u_1 impinge on the mass m_2 which is supported by the rod AB of uniform cross-section A and length l. Let v be the velocity of both masses during impact. Then from equation (3), page 145, we have



$$v = \frac{m_1 u_1}{m_1 + m_2},$$

and the work in foot-pounds necessary to bring the combined masses to rest is

$$W = \frac{1}{2g}(m_1 + m_2)v^2 = \frac{u_1^2}{2g} \cdot \frac{m_1^2}{m_1 + m_2}$$

$$= \frac{m_1^2 h}{m_1 + m_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\frac{u_1^2}{2g} = h$ is the height of fall of m_1.

This work is equal to the work of stretching or compressing the rod, or equal to $\frac{1}{2}F\lambda$, where F is the stress of impact and λ the strain, since work is equal to one half the product of stress and strain (page 281, Vol. II, *Statics*). But (page 281, Vol. II, *Statics*) within the limit of elasticity we have

$$F = \frac{EA\lambda}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where E is the coefficient of elasticity. Hence

$$\frac{1}{2}F\lambda = \frac{EA\lambda^2}{2l} = \frac{m_1^2 h}{m_1 + m_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$\lambda = \sqrt{\frac{m_1^2}{m_1 + m_2} \cdot \frac{2lh}{EA}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (4) we can find the strain λ of the rod caused by the impact. If the rod is strained up to the limit of elasticity S_e, we have from (2), by putting $F = S_e A$,

$$\lambda = \frac{S_e l}{E}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and hence from (3)

$$\frac{S_e^2}{2E} \cdot Al = \frac{m_1^2 h}{m_1 + m_2}.$$

But Al is the volume of the rod V. The velocity of impact

$$u_1 = \sqrt{2gh}$$

which is necessary to strain the rod up to the limit of elasticity is then given by

$$h = \frac{m_1 + m_2}{m_1^2} \cdot \frac{S_e^2}{2E} \cdot V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The quantity $\frac{S_e^2}{2E}$ is the *coefficient of resilience* (page 282, Vol. II, *Statics*).

We see from (6) that the greater the volume or mass of the rod the greater the blow it can bear. Hence the mass of bodies subjected to impact should be made as great as possible.

Since m_1 and m_2 fall during impact through the distance λ, we have more correctly

$$W = \frac{m_1^2 h}{m_1 + m_2} + (m_1 + m_2)\lambda;$$

and hence, instead of (6), we have

$$h = \frac{m_1 + m_2}{m_1^2} \cdot \frac{S_e^2}{2E} \cdot V - \frac{(m_1 + m_2)^2}{m_1^2} \cdot \frac{S_e l}{E}. \quad . \quad . \quad . \quad . \quad (7)$$

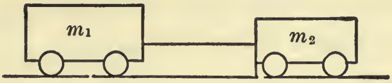
If, finally, we wish to take into consideration the mass m_3 of the rod, we have, since its centre of mass moves through the distance $\frac{1}{2}\lambda$,

$$W = \frac{m_1^2 h}{m_1 + m_2 + m_3} + \left(m_1 + m_2 + \frac{1}{2}m_3\right)\lambda;$$

and hence, instead of (6), we have

$$h = \frac{m_1 + m_2 + m_3}{m_1^2} \cdot \frac{S_e^2}{2E} \cdot V - \frac{\left(m_1 + m_2 + m_3\right)\left(m_1 + m_2 + \frac{1}{2}m_3\right)}{m_1^2} \cdot \frac{S_e l}{E}. \quad (8)$$

If a mass m_1 moving with a velocity u_1 puts in motion another mass m_2 by means of a chain or rope, we have in the same way for



the velocity of both bodies during impact

$$v = \frac{m_1 u_1}{m_1 + m_2},$$

and the work in foot-pounds expended in stretching the rope or chain is

$$W = \frac{1}{2g} m_1 u_1^2 - \frac{1}{2g}(m_1 + m_2)v^2 = \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{u_1^2}{2g} = \frac{m_1 m_2}{m_1 + m_2} \cdot h,$$

where $h = \frac{u_1^2}{2g}$ = the height due to the velocity.

We have then, if the chain or rope is stretched to the limit of elasticity S_e,

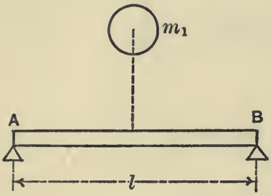
$$\frac{S_e^2}{2E} \cdot Al = \frac{m_1 m_2}{m_1 + m_2} h,$$

where A is the cross-section and l the length of the chain or rope. Hence

$$h = \frac{m_1 + m_2}{m_1 m_2} \cdot \frac{S_e^2}{2E} \cdot V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where V is the volume of the chain or rope.

[**Impact of Beams.**—Let a mass m_1 fall from a height h upon a beam AB of uniform cross-section A and span l, supported at the ends.



Let δ be the density of the beam and V its volume. Then

$$V = Al,$$

and the mass m_2 of the beam is

$$m_2 = \delta Al = \delta V. \quad . \quad . \quad . \quad . \quad (1)$$

Let the mass m_1 strike the beam at the centre and let the equivalent mass of the beam concentrated at the centre be nm_2. Let the velocity of the combined masses m_1 and nm_2 at the centre of the beam during impact be v_c, and let the velocity of m_1 before impact be u_1. Then we have from equation (3), page 145,

$$v_c = \frac{m_1 u_1}{m_1 + nm_2},$$

and the work in foot-pounds necessary to bring the combined masses to rest is

$$W = (m_1 + nm_2)\frac{v_c^2}{2g} = \frac{m_1^2}{m_1 + nm_2} \cdot \frac{u_1^2}{2g} = \frac{m_1^2 h}{m_1 + nm_2}, \quad . \quad . \quad (2)$$

where $h = \frac{u_1^2}{2g}$ is the height of fall of m_1.

Let P be the pressure during impact and Δ the deflection at the centre. Then the work of deflection is $\frac{1}{2}P\Delta$, and we have

$$\frac{m_1^2 h}{m_1 + nm_2} = \frac{1}{2}P\Delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the beam is strained up to the limit of elasticity S_e, we have, if e is the distance of the most remote fibre from the neutral axis, from page 326, Vol. II, *Statics*,

$$\frac{S_e I}{e} = \frac{Pl}{4}, \quad \text{or} \quad P = \frac{4S_e I}{el}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where I is the moment of inertia of the cross-section of the beam, with reference to the neutral axis.

From page 335, Vol. II, *Statics*, we have

$$\Delta = \frac{Pl^3}{48EI} = \frac{S_e l^2}{12eE}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where E is the coefficient of elasticity.

From (4) and (5) we have

$$\frac{1}{2}P\varDelta = \frac{S_e^2 Il}{6e^2 E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We can substitute this value of $\frac{1}{2}P\varDelta$ in (3). It remains to find the equivalent mass nm_2 of the beam, considered as concentrated at the centre.

Let the velocity of any point of the beam during impact be v and its deflection y. Then we have

$$\frac{v}{v_c} = \frac{y}{\varDelta}, \quad \text{or} \quad v = \frac{y}{\varDelta}v_c.$$

The mass of an indefinitely small portion of the beam of length dx is $\delta A dx$, and its energy is $\delta A dx\frac{v^2}{2g}$. If then nm^2 is the equivalent mass at the centre, we have

$$nm_2\frac{v_c^2}{2g} = \int_0^l \delta A dx\frac{v^2}{2g} = \int_0^l \delta A dx\frac{y^2}{\varDelta^2}\cdot\frac{v_c^2}{2g},$$

or

$$nm_2 = \frac{\delta A}{\varDelta^2}\int_0^l y^2 dx.$$

From page 335, Vol. II, *Statics*, we have

$$y = \frac{P}{12EI}\left(x^3 - \frac{3}{4}l^2 x\right).$$

Hence we have

$$nm_2 = \frac{16\delta A}{l^6}\int_0^l\left(x^6 dx - \frac{3}{2}l^2 x^4 dx + \frac{9}{16}l^4 x^2 dx\right).$$

Performing the integration, we obtain

$$nm_2 = \frac{17}{35}\delta Al = \frac{17}{35}m_2, \quad \text{or} \quad n = \frac{17}{35}.$$

We have then from (3) and (6)

$$\frac{m_1{}^2 h}{m_1 + \frac{17}{35}m_2} = \frac{S_e Il}{6Ee^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If, for instance, the cross-section of the beam is a rectangle of breadth b and depth d, we have (page 277, Vol. II, *Statics*) $I = \frac{1}{12}bd^3$ and $e = \frac{1}{2}d$. Hence for this case

$$\frac{m_1{}^2 h}{m_1 + \frac{17}{35}m_2} = \frac{S_e^2 bdl}{18E},$$

or putting $bdl = V =$ the volume of the beam and $V\delta = m_2$, we have for the height of fall necessary to strain the beam to the limit of elasticity

$$h = \frac{S_e^2 m_2\left(m_1 + \frac{17}{35}m_2\right)}{18\delta m_1{}^2 E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

EXAMPLES.

(1) *An inelastic body of mass* $m_1 = 50$ *pounds moving with a velocity of* $u_1 = 7$ *feet per second impinges upon another of mass* $m_2 = 30$ *pounds moving in the same direction with a velocity of* $u_2 = 3$ *feet per sec. Find the velocity with which the two move on together after collision.*

Ans. $5\frac{1}{2}$ ft. per sec.

(2) *In order to cause an inelastic body weighing* 120 *pounds to change its velocity from* $1\frac{1}{2}$ *to* 2 *feet per sec., we let an inelastic body weighing* 50 *pounds strike it. Find the velocity of the latter body.*

Ans. 3.2 ft. per sec.

(3) *Two inelastic masses of* 3 *and* 5 *tons impinge with velocities of* 4 *and* 5.5 *ft. per sec. respectively. Find their final velocity when they are moving in the same and in opposite directions.*

Ans. $4\frac{15}{16}$ feet per sec.; $1\frac{15}{16}$ ft. per sec. in the direction of the larger velocity.

(4) *Two inelastic particles of* 3 *lbs. and* 1 *oz. are moving in opposite directions and impinge. The first has a velocity of* $3\frac{1}{2}$ *and the latter of* 9 *ft. per sec. In what direction do they move after impact?*

Ans. In the direction of the first with a velocity of 21/31 ft. per sec.

(5) *An inelastic particle whose mass is* 16 *lbs. moving with a velocity of* 25 *miles an hour impinges on another moving in the opposite direction. The two come to rest. If the mass of the latter were* 28 *lbs., find its velocity. If the velocity of the latter were* 66 *ft. per sec., find its mass.*

Ans. $14\frac{2}{7}$ miles per hour; $8\frac{8}{9}$ lbs.

(6) *A number of inelastic balls of masses* m_1, m_2, m_3, *etc.*, m_n, *lie on a straight line at rest. If the first have a velocity of* u_1 *towards the others what will be the ultimate velocity of the balls?*

Ans. $v_n = \dfrac{m_1 u_1}{m_1 + m_2 + \ldots m_n}$.

(7) *If in the preceding example the initial velocities are* u_1, u_2, $u_3 \ldots u_n$, *find the ultimate velocity.*

Ans. $v_n = \dfrac{m_1 u_1 + m_2 u_2 + \ldots m_n u_n}{m_1 + m_2 + \ldots m_n}$.

(8) *A shot of* 600 *lbs. is fired from a* 10-*ton gun with a velocity of* 1000 *feet per sec. If the mass of the powder be neglected, find the velocity of recoil.*

Ans. $26\frac{11}{14}$ ft. per sec.

(9) *An* 1800-*lb. shot moving with a velocity of* 2000 *ft. per sec. strikes a* 10-*ton plate, passes through it and goes on with a velocity of* 400 *ft. per sec. If the plate be free to move, find its velocity.*

Ans. $128\frac{4}{7}$ ft. per sec.

(10) *Two perfectly elastic balls weighing* 10 *lbs. and* 16 *lbs. collide with the velocities* 12 *and* 6 *feet per sec. Find the loss and gain of velocity, and the velocities after collision, if the velocities are in the same and in opposite directions.*

Ans. In the first case the final velocities are $v_1 = +4\frac{8}{13}$ and $v_2 = +10\frac{8}{13}$ ft. per sec. The first body loses then $7\frac{5}{13}$ and the other gains $4\frac{8}{13}$ ft. per sec.

In the second case the final velocities are $v_1 = -10\frac{2}{13}$ and $v_2 = +7\frac{11}{13}$ ft. per sec. Each body then rebounds with these velocities. The first body loses $22\frac{2}{13}$ and the other gains $13\frac{8}{13}$ ft. per sec.

(11) *A number of perfectly elastic balls of masses $m_1, m_2, m_3 \ldots m_n$ lie on a straight line at rest. If the first have a velocity of u_1 towards the others, find the velocities after impact.*

Ans. The velocity of the first is

$$v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2}.$$

The velocity of any intermediate ball is

$$v_n = \frac{2^{n-1}m_1 \,.\, m_2 \ldots m_{n-1}(m_n - m_{n+1})u_1}{(m_1 + m_2)(m_2 + m_3) \ldots (m_n + m_{n+1})}.$$

The velocity of the last ball is

$$v_n = \frac{2^{n-1}m_1 \,.\, m_2 \ldots m_{n-1}u_1}{(m_1 + m_2)(m_2 + m_3) \ldots (m_{n-1} + m_n)}.$$

(12) *In the preceding example let there be four balls, the mass of the first m_1, of the second $m_2 = am_1$, of the third $m_3 = am_2 = a^2m_1$, of the fourth $m_4 = am_3 = a^3m_1$.*

Ans. $v_1 = \frac{1-a}{1+a}u_1$; $v_2 = \frac{2}{1+a}u_1$; $v_3 = \frac{4}{(1+a)^2}u_1$; $v_4 = \frac{8}{(1+a)^3}u_1$.

If, for example, the mass of each ball is one half that of the preceding, we have

$$v_1 = \frac{1}{3}u_1, \quad v_2 = \frac{4}{3}u_1, \quad v_3 = \frac{16}{9}u_1, \quad v_4 = \frac{64}{27}u_1.$$

(13) *If in a machine 16 impacts per minute take place between the two inelastic masses $m_1 = 1000$ lbs. and $m_2 = 1200$ lbs. moving with velocities $u_1 = 5$ and $u_2 = 2$ ft. per sec., find the loss of energy.*

Ans. $\frac{16}{60} \cdot \frac{(5-2)^2}{2g} \cdot \frac{1000 \times 1200}{2200} = 20.3$ ft.-lbs. per sec.

(14) *If two trains $m_1 = 120000$ lbs. and $m_2 = 160000$ lbs. come into collision with the opposite velocities $u_1 = 20$ and $u_2 = 15$ ft. per sec., find the loss of energy which is expended in the destruction of the cars, considering them as inelastic.*

Ans. $\frac{(20+15)^2}{2g} \cdot \frac{120000 \times 160000}{280000} = 1302000$ ft.-lbs.

(15) *If an iron sledge of mass $m_1 = 50$ lbs., $l_1 = 6$ inches long and $A_1 = 4$ sq. inches area of face, strikes an immovable lead plate $l_2 = 1$ inch thick and $A_2 = 2$ sq. inches area, with a velocity $u_1 = 50$ ft. per sec., find the compression of the sledge and plate and the force of impact, taking $E_1 = 29000000$, $E_2 = 700000$ lbs. per sq. in.*

Ans. We have for the hardness

$$H_1 = \frac{A_1E_1}{l_1} = \frac{4 \times 29000000}{\frac{1}{2}} = 232000000 \text{ lbs. per ft.};$$

$$H_2 = \frac{A_2E_2}{l_2} = \frac{2 \times 700000}{\frac{1}{12}} = 16800000 \text{ lbs. per ft.}$$

The total compression is

$$\lambda_1 + \lambda_2 = 50\sqrt{\frac{50}{g}\cdot\frac{232000000 + 16800000}{389760000000000}} = 0.0158 \text{ ft.} = 0.19 \text{ inch.}$$

The force of impact is

$$F = \frac{H_1 H_2}{H_1 + H_2}(\lambda_1 + \lambda_2) = 247516 \text{ lbs.},$$

and

$$\lambda_1 = \frac{F}{H_1} = 0.00107 \text{ ft.} = 0.0128 \text{ in.},\quad \lambda_2 = \frac{F}{H_2} = 0.0147 \text{ ft.} = 0.177 \text{ in.}$$

(16) *In the preceding example consider the sledge as perfectly elastic and the plate as inelastic.*

Ans. We have for the loss of velocity of the sledge

$$u_1 - v_1 = u_1\left[1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right] = 50\left[1 + \sqrt{\frac{16800000}{248800000}}\right] = 63 \text{ ft. per sec.}$$

Hence $v_1 = -13$ ft. per sec., or the sledge rebounds with this velocity.

(17) *What will be the velocities of two steel plates after impact if the velocities before impact are* $u_1 = 10$ *and* $u_2 = -6$ *ft. per sec. and the masses* $m_1 = 30$ *lbs.,* $m_2 = 40$ *lbs., taking* $e = \frac{5}{9}$?

Ans. $u_1 - v_1 = (10 + 6)\frac{40}{70}\left(1 + \frac{5}{9}\right) = 14.22$ ft. per sec.

Hence $v_1 = -4.22$ ft. per sec., or the first plate rebounds with this velocity.

$$v_2 - u_2 = (10 + 6)\frac{30}{70}\left(1 + \frac{5}{9}\right) = 10.665 \text{ ft. per sec.}$$

Hence $v_2 = +4.665$ ft. per sec., or the second plate rebounds with this velocity.

(18) *Two balls* $m_1 = 30$ *lbs.,* $m_2 = 50$ *lbs. strike each other with the velocities* $u_1 = 20$ *and* $u_2 = 25$ *ft. per sec., making the angles with the line of impact* $\alpha_1 = 21° 35'$ *and* $\alpha_2 = 65° 20'$. *Find the velocities after impact if the bodies are inelastic.*

Ans. $u_1 \sin\alpha_1 = 7.357$ ft. per sec., $u_2 \sin\alpha_2 = 22.719$ ft. per sec.;
$u_1 \cos\alpha_1 = 18.598$ " " " $u_2 \cos\alpha_2 = 10.433$ " " "

Hence

$$v_1 = 18.598 - (18.598 - 10.433)\frac{50}{80} = 13.495 \text{ ft. per sec.};$$

$$v_2 = 10.433 + (18.598 - 10.433)\frac{30}{80} = 13.495 \text{ `` `` ``}$$

The resulting velocities are then

$$w_1 = \sqrt{13.495^2 + 7.357^2} = 15.37 \text{ ft. per sec.};$$

$$w_2 = \sqrt{13.495^2 + 22.719^2} = 26.42 \text{ `` `` ``}$$

making the angles β_1 and β_2 with the line of impact given by

$$\tan\beta_1 = \frac{7.357}{13.495},\quad \text{or}\quad \beta_1 = 28° 36';$$

$$\tan\beta_2 = \frac{22.719}{13.495},\quad \text{or}\quad \beta_2 = 59° 17'.$$

(19) *A billiard-ball strikes the cushion with a velocity of* $u_1 = 15$ *ft. per sec., the angle of incidence being* $\alpha_1 = 45°$. *If* $e = 0.55$ *and the coefficient of friction is* $\mu = 0.2$, *find the motion after impact.*

Ans. The velocity after impact along the line of impact is

$$v_1 = -eu_1 \cos \alpha = -0.55 \times 15 \cos 45° = -5.833 \text{ ft. per sec.}$$

The velocity parallel to the cushion is

$$u_1 \sin \alpha_1 - \mu u_1 \cos \alpha_1 (1 + e) = 7.319 \text{ ft. per sec.}$$

Hence the angle of reflection β_1 is given by

$$\tan \beta_1 = \frac{7.319}{5.833}, \quad \text{or} \quad \beta_1 = 51° \, 27',$$

and the velocity after impact is

$$w_1 = \frac{5.833}{\cos 51° \, 27'} = 9.36 \text{ ft. per sec.}$$

The ball also acquires the velocity of rotation about a vertical through its centre of mass of

$$\frac{5}{2}\mu \times 1.55 u_1 \cos 45° = 8.22 \text{ ft. per sec.}$$

If the ball rolls on the table without sliding, it has, besides its velocity $u_1 = 15$ ft. per sec. of translation, an equal velocity at the circumference, and this can be resolved into the components $u_1 \sin \alpha_1 = 10.607$ ft. per sec. about an axis normal to the cushion, and a component $u_1 \cos \alpha_1 = 10.607$ ft. per sec. about an axis parallel to the cushion. The first component is unchanged by friction. The second component becomes changed by friction to

$$u_1 \cos \alpha_1 - \frac{5}{2}\mu(1 + e) u_1 \cos \alpha_1 = 10.607 - 8.22 = 2.387 \text{ ft. per sec.}$$

(20) *A maul whose weight is* 120 *lbs. falls upon a mass of earth from a height of* 4 *ft., and the earth is compressed one fourth of an inch by the last blow. The cross-section of the maul is* 5/4 *sq. ft. What weight will the earth sustain safely, taking a factor of safety of* 10?

Ans. $F = \frac{mh}{s} = \frac{120 \times 4}{\frac{1}{48}} = 23040$ lbs. The force per square foot is then $\frac{23040}{\frac{5}{4}} = 18432$ lbs. per sq. ft. Taking 10 for a factor of safety, we have 1843.2 lbs. per sq. ft.

(21) *A pile whose cross-section is* 1 *sq. ft. and length* 25 *ft. and mass* 1200 *lbs. is driven by the last tally of ten blows of a ram weighing* 2000 *lbs. and falling* 6 *ft.,* 2 *inches deeper. Taking the coefficient of elasticity of the pile* $E_2 = 1560000$ *lbs. per sq. in., find the weight the pile can safely sustain for a factor of safety of* 6.

Ans. The hardness for the pile is

$$H_2 = \frac{A_2 E_2}{l_2} = \frac{144 \times 1560000}{25} = 8985600 \text{ lbs. per ft.}$$

The mean depth of penetration for one blow is $\frac{2}{10}$ inch $= \frac{1}{60}$ ft. Hence

$$= \frac{2000 \times 6}{\frac{1}{60} + \sqrt{\frac{2000 \times 1200 \times 6}{2(2000 + 1200) \times 8985600}}} = 378000 \text{ lbs.}$$

For a factor of safety of 6 we have $F = \frac{378000}{6} = 63000$ lbs.

(22) *The two opposite suspension-rods of a suspension-bridge support a constant weight of* 5000 *pounds, which is increased by* 6000 *pounds by a passing wagon. The coefficient of resilience* $\frac{S_e^2}{2E}$ *of wrought iron is* 7 *inch-lbs. per cubic inch* (*page* 282, *Vol. II, Statics*). *The length of the rods is* 200 *inches and their cross-section* 1.5 *sq. in. Find the height of fall to stretch the rods to the limit of elasticity.*

Ans. $h = \frac{(5000 + 6000)7 \times 200 \times 1.5 \times 2}{6000^2} = 1.28$ inches.

If, then, the wagon passes over an obstacle 1.3 inches high, the rods are in danger of being stretched beyond the elastic limit.

(23) *Find the height from which a mass of* 200 *lbs. must fall in order that, striking the centre of a plate of cast iron* 36 *inches long,* 12 *inches wide and* 3 *inches thick, supported at both ends, it may bend it to the elastic limit.*

Ans. If we take the coefficient of resilience (page 282, Vol. II, *Statics*)

$$\frac{S_e^2}{2E} = 1.2 \text{ inch-lbs. per cubic inch,}$$

we have, since δ for cast iron is about 0.259 lb. per cubic inch and $V = 12 \times 3 \times 36 = 1296$ cubic inches,

$$m_2 = V\delta = 335.7 \text{ pounds.}$$

We have then for the height of fall

$$h = \frac{S_e^2 m_2 \left(m_1 + \frac{17}{35} m_2\right)}{18\delta m_1^2 E} = 1.57 \text{ inches.}$$

KINETICS OF A RIGID BODY.

CHAPTER I.

ROTATION ABOUT A FIXED AXIS.

ROTATION. PLANE OF ROTATION. IMPRESSED AND EFFECTIVE FORCES. D'ALEMBERT'S PRINCIPLE. MOMENT OF EFFECTIVE FORCES. MOMENT OF INERTIA OF A BODY. MOMENT OF MOMENTUM. MOMENT OF IMPULSE. KINETIC ENERGY OF ROTATION. ANALOGY BETWEEN EQUATIONS FOR ROTATION AND RECTILINEAR MOTION. REDUCTION OF MOMENT OF INERTIA. REDUCTION OF MASS. MOMENT OF INERTIA FOR RECTANGLE, ELLIPSE OR ELLIPSOID. RADIUS OF GYRATION. CENTRE OF PERCUSSION. COMPOUND PENDULUM. IMPACT OF AN OSCILLATING BODY. BALLISTIC PENDULUM. ECCENTRIC IMPACT. TORSION - PENDULUM. GENERAL FORMULAS FOR ROTATION ABOUT A FIXED AXIS.

Rotation.—We have proved (Vol. II, *Statics*, page 83) that when a body is acted upon by any forces, *the motion of the centre of mass is the same as if all the mass were collected at the centre of mass, and all the forces were applied at that point parallel to their actual directions.*

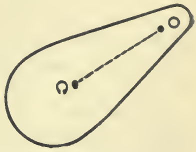
Thus far we have considered the motion of bodies without reference to rotation. So far, then, as translation only is concerned, we have considered bodies as if they were particles and thus have treated of **Kinetics of a Particle** or **Translation.**

But a body may have a motion of rotation only about a fixed axis, and in the present Chapter we shall discuss such motion.

Plane of Rotation—Centre of Rotation or Point of Suspension.—Let a rigid body rotate about a fixed axis, and let C be the centre of mass.

If we pass a plane through C perpendicular to the axis, this plane is called the **plane of rotation,** and the intersection O of the axis with this plane is called the **centre of rotation** or **point of suspension.** All the external forces acting upon the body must reduce to a single resultant force in the plane of rotation.

Impressed and Effective Forces.—The external forces acting upon any body we call **impressed forces.** Thus any force acting upon the body due to the action of some other body is an *impressed* force, or force *impressed* upon the body.



Every particle of a rigid body, since it must partake of the motion of the body, must be considered as acted upon by all the impressed forces, transferred to the particle without change in magnitude or direction.

Every particle of a rigid body is also acted upon by all the other particles adjacent to it, that is, by the *internal* or *molecular forces.*

If now we consider any one particle *by itself*, uninfluenced therefore by the impressed or by the molecular forces, there is a certain force which would make that particle move at any instant precisely as it did move at that instant when it formed a part of the rigid body. Thus, if its mass is m and its acceleration, when part of the rigid body, was f in a certain direction, then this force is mf. This we call the **effective force** on the particle. Each and every particle of the body considered as acted upon at any instant solely by its own effective force at that instant will move as part of the rigid body at that instant.

D'Alembert's Principle.—We distinguish then external or **impressed forces** acting upon the rigid body, internal or **molecular forces** acting between the particles, and an **effective force** on each particle which, acting by itself, would make each particle move at any instant as part of the rigid body.

Let us denote the resultant of the impressed forces by F and the resultant of the molecular forces by R. Then since a body cannot change its own motion, F is the cause of change of motion. Also F must be the resultant of R and all the effective forces.

But since by Newton's third law (page 36) action and reaction between any two particles are equal and opposite, the internal or molecular forces between the particles form a system of forces in equilibrium and hence their resultant R is zero.

Therefore F must be the resultant of all the effective forces. Hence if these effective forces are *reversed in direction* they will form with F a system of forces in equilibrium.

We have then the following principle:

The impressed forces acting upon a body and the reversed effective forces for all the particles of a body constitute a system of forces in equilibrium.

This principle, stated by *D'Alembert* in 1742, is known as **D'Alembert's Principle.** It reduces any dynamic problem to one of static equilibrium between actual ("*impressed*") forces and fictitious ("*reversed effective*") forces.

Thus suppose the resultant of all the forces acting upon a body is a uniform force F, and that the body has a motion of translation in the direction of F. Then every particle of the body of mass m has the same acceleration f in the direction of F. The sum of the effective forces is then $\Sigma mf = f\Sigma m = Mf$, where M is the mass of the body. Reversing the direction of these forces, we have for equilibrium

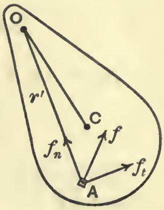
$$F - Mf = 0 \quad \text{or} \quad F = Mf,$$

which is the equation of force (page 2).

Also F must act at the centre of the parallel forces mf or at the centre of mass.

In order to apply D'Alembert's principle to a rotating body, we must evidently first be able to find the *sum of the moments of the effective forces* with reference to the axis of rotation.

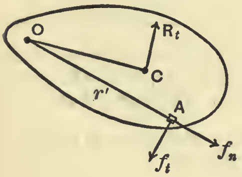
Moment of the Effective Forces—Rotation.—Let a rigid body rotate about a fixed axis at O, perpendicular to the plane of the paper. Let C be the centre of mass. Pass a plane through the centre of mass C *perpendicular to the axis.*



Since the motion of the centre of mass is the same as if all the mass of the body were concentrated at that point and all the impressed forces acted at that point (page 167), all the components of the impressed forces which cause rotation must act in planes parallel to the plane through C perpendicular to the axis. This plane (page 167) is the **plane of rotation**, and its intersection O with the axis is the **centre of rotation**.

Now consider any particle of the body at A, distant $OA = r'$ from the axis, and let the mass of this particle be m and its linear acceleration be f. This particle moves in a circle whose radius is r' and whose plane is parallel to the plane of rotation. Its acceleration f can be resolved into the tangential acceleration $f_t = r'\alpha$, where α is the angular acceleration of the body, and the normal acceleration $f_n = r'\omega^2$, where ω is the angular velocity of the body . The effective force is then mf and its tangential component is $mf_t = mr'\alpha$ and its normal component is $mf_n = mr'\omega^2$. We have in like manner for each and every particle of the body a tangential effective force in the direction of motion $mr'\alpha$, and a deflecting effective force $mr'\omega^2$, where r' is the distance of the particle from the axis.

Suppose these effective forces on each particle *reversed in direction*, so that f_n acts away from the axis and f_t opposite to the direction of motion.



Then by D'Alembert's principle the impressed forces and all these reversed effective forces must constitute a system of forces in equilibrium.

Since the axis is fixed, the algebraic sum of the components of all the forces in each of any three rectangular directions must be zero. The first of the conditions for equilibrium for a system of forces acting upon a rigid body (Vol. II, *Statics*, page 85) is therefore necessarily fulfilled.

In order that the second condition (Vol. II, *Statics*, page 85) may be fulfilled we must have the algebraic sum of the moments of all the forces about the axis equal to zero.

Since the axis is fixed, the algebraic sum of the moments of the components of all the forces at right angles to the plane of rotation is zero. It remains to consider the components parallel to the plane of rotation.

Let the components of the impressed forces parallel to the plane of rotation be F_1, F_2, F_3, etc., and their lever-arms with reference to the axis be p_1', p_2', p_3', etc. Then $\Sigma Fp'$ is the algebraic sum of the moments of the components of the impressed forces parallel to the plane of rotation.

The moment of the effective force mf for a particle at A is equal to the algebraic sum of the moments of its components mf_t and mf_n. But mf_n passes through the axis and its moment is then zero. Hence the moment of mf about the axis is equal to the moment of $mf_t = mr'\alpha$, or is equal to $mr'\alpha \times r' = mr'^2\alpha$. The sum of the moments of all the effective forces about the axis is then $\Sigma mr'^2\alpha$.

Reversing them, we have for equilibrium by D'Alembert's principle

$$\Sigma Fp' - \Sigma mr'^2\alpha = 0, \quad \text{or} \quad \Sigma mr'^2\alpha = \Sigma Fp',$$

or, since for a rigid body α is the same for every particle,

$$\alpha\Sigma mr'^2 = \Sigma Fp'.$$

Moment of Inertia of a Body.—The mass m of a particle multiplied by the square of its distance r from a point, axis or plane is called the **moment of inertia of the particle** with respect to that point, axis or plane.

The sum of the products mr'^2 for all the particles of a body, or $\Sigma mr'^2$, is called the **moment of inertia of the body** with reference to the axis at O.* The moment of inertia of a body with reference to any axis we denote in general by I', and with reference to an axis *through the centre of mass* by I.

We have then, from the preceding article, adopting this notation,

$$I'\alpha = \Sigma Fp'. \qquad \text{(I)}$$

That is, when a body rotates about any axis, *the algebraic sum of the moments of all the impressed forces with reference to that axis is equal to the moment of inertia*† *of the body with reference to that axis, multiplied by the angular acceleration of the body.*

The product $I'\alpha$ evidently gives the moment in *poundal-feet.* For pound-feet we must divide by g.

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[*] If a is an elementary *area*, Σma^2 is the moment of inertia *of the area.* (Vol. II, *Statics*, page 272.) Hence, if δ is the surface density, the values of I given in Vol. II, multiplied by δ, will give the moments of inertia for *material areas.*

[†] The term "moment of inertia" is due to Euler. Euler used the term "inertia" as synonymous with *mass.* The equation of force

$$F = mf$$

would then be read, in the terminology of Euler,

Force = "*inertia*" × linear *acceleration.*

In the equation

$$\Sigma Fp = \alpha\Sigma mr^2,$$

since ΣFp is the *moment* of the resultant force and α is the angular acceleration, Euler called the quantity Σmr^2 "moment of inertia," and thus obtained the analogous expression for rotation,

"moment *of force* = moment *of inertia* × angular *acceleration.*"

The term "moment of inertia" in modern scientific terminology is an improper expression. Inertia is a property of matter like color or hardness, and we cannot properly speak of "moment of inertia" any more than of moment of color or hardness. The term "second moment of mass" would more correctly describe the product mr^2, and has been indeed used. The expression "moment of inertia," however, has become firmly established by long usage.

The student, while using it, then, should consider it simply as a name for a quantity, mr^2 or Σmr^2, which occurs so frequently in dynamic problems that it is convenient to give it a special name.

Moment of Momentum.—We have called the product mv of the mass m of a body or particle by its linear velocity v the *momentum* of the body or particle (page 32), and we have illustrated by examples, page 37, its significance and use.

If the particle is moving in a circle of radius r' its velocity $v = r'\omega$, where ω is the angular velocity. The momentum is then $mv = mr'\omega$. If we multiply this by r' we have mvr' = the *moment of the momentum of the particle* $= mr'^2\omega$. If we take the sum for all the particles of a rotating body we have $\Sigma mvr' =$ *moment of momentum* $= \Sigma mr'^2\omega$, or, since ω is the same for all particles,

$$\Sigma mvr' = \textit{moment of momentum} = \omega\Sigma mr'^2 = I'\omega. \quad . \quad . \quad \text{(II)}$$

That is, *when a body rotates about any axis with the angular velocity* ω, *the algebraic sum of the moments of momentum for all the particles is equal to the moment of inertia of the body with reference to that axis, multiplied by the angular velocity of the body.*

Moment of Impulse.—We have seen (page 31) that the product of a uniform force F by its time of action t, or Ft, is called the impulse of a force, and we have, denoting the impulse by ϕ,

$$\phi = Ft = m(v - v_1),$$

where m is the mass of a particle and v and v_1 its final and initial velocities.

For a particle moving in a circle of radius r' we have $v = r'\omega$ and $v_1 = r'\omega_1$, where ω and ω_1 are the final and initial angular velocities. If we multiply by r' we have then $\phi r' =$ *moment of impulse* $= mr'^2(\omega - \omega_1)$. If we take the sum for all the particles of a rotating body we have

$$\Sigma\phi r' = \text{moment of impulse} = (\omega - \omega_1)\Sigma mr'^2 = I'(\omega - \omega_1). \quad \text{(III)}$$

That is, when a body rotates about any axis and under the action of a tangential force of constant magnitude its angular velocity changes from ω_1 to ω, the moment of the impulse is equal to the change of moment of momentum; or,

Change of moment of momentum is equal to the moment of the impulse of the tangential force of constant magnitude which causes it. (Compare page 34.)

Kinetic Energy of a Rotating Body.—The kinetic energy of a particle of mass m and velocity v is $\frac{1}{2}mv^2$ (page 56). If the particle moves in a circle of radius r' we have $v = r'\omega$, where ω is the angular velocity, and hence the kinetic energy is $\frac{1}{2}mr'^2\omega^2$. If we take the sum for all the particles of a rotating body we have

$$\text{Kinetic energy} = \frac{1}{2}\omega^2\Sigma mr'^2 = \frac{1}{2}I'\omega^2. \quad . \quad . \quad . \quad \text{(IV)}$$

That is, *the kinetic energy of a body rotating about any axis with the angular velocity* ω *is equal to one half the moment of inertia of the body with reference to that axis multiplied by the square of the angular velocity.*

The product $\frac{1}{2}I'\omega^2$ evidently gives the energy in *foot-poundals*. For foot-pounds we must divide by g.

Analogy between the Equations for Rotation and Rectilinear Motion.—The student should not fail to note the analogy between the equations (I), (II), (III) and (IV) for rotation and the corresponding equations for rectilinear motion, and to recognize the part played by the quantity $I' = \Sigma mr'^2$, which we have called the "moment of inertia" of the rotating body.

Thus for rectilinear motion

$$F = mf, \quad \text{or} \quad \text{Force in poundals}$$

$$= \text{(mass)} \times \textit{linear} \text{ acceleration}, \quad . \; . \; . \; . \; . \; . \quad (1)$$

while for rotation

$$\Sigma Fp' = I'\alpha, \quad \text{or} \quad \textit{moment} \text{ of Force in poundals}$$

$$= \begin{pmatrix}\text{moment}\\ \text{of}\\ \text{inertia}\end{pmatrix} \times \textit{angular} \text{ acceleration.} \quad . \; . \; . \quad (\text{I})$$

For rectilinear motion

$$M = mv, \quad \text{or} \quad \text{momentum}$$

$$= \text{(mass)} \times \textit{linear} \text{ velocity}, \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

while for rotation

$$\Sigma mvr' = I'\omega, \quad \text{or} \quad \textit{moment} \text{ of momentum}$$

$$= \begin{pmatrix}\text{moment}\\ \text{of}\\ \text{inertia}\end{pmatrix} \times \textit{angular} \text{ velocity.} \quad . \; . \; . \; . \quad (\text{II})$$

For rectilinear motion

$$\phi = m(v - v_1), \quad \text{or} \quad \text{impulse}$$

$$= \text{(mass)} \times \text{change of } \textit{linear} \text{ velocity}, \quad . \; . \; . \quad (3)$$

while for rotation

$$\Sigma \phi r' = I'(\omega - \omega_1), \quad \text{or} \quad \textit{moment} \text{ of impulse}$$

$$= \begin{pmatrix}\text{moment}\\ \text{of}\\ \text{inertia}\end{pmatrix} \times \text{change of } \textit{angular} \text{ velocity.} \quad (\text{III})$$

For rectilinear motion

$$E = \frac{1}{2}mv^2, \quad \text{or} \quad \text{kinetic energy in foot-poundals}$$

$$= \frac{1}{2}\text{(mass)} \times \text{square of } \textit{linear} \text{ velocity}, \quad . \; . \; . \quad (4)$$

while for rotation

$$E = \frac{1}{2}I'\omega^2, \quad \text{or} \quad \text{kinetic energy in foot-poundals}$$

$$= \frac{1}{2}\begin{pmatrix}\text{moment}\\ \text{of}\\ \text{inertia}\end{pmatrix} \times \text{square of } \textit{angular} \text{ velocity.} \quad (\text{IV})$$

We see then that in the equations for force, momentum and impulse for rectilinear motion, if we replace mass m by moment of inertia I' and *linear* acceleration and velocity f and v_1 by *angular* acceleration or velocity α and ω_1, we obtain *moment* of force, momentum and impulse for rotation. Also in the equation for kinetic energy for rectilinear motion, if we replace mass by moment of inertia and *linear* by *angular* velocity, we obtain kinetic energy for rotation. These formulas, together with D'Alembert's principle, reduce every problem of rotation to one of static equilibrium.

Reduction of Moment of Inertia.—If I is the moment of inertia of a body with reference to an axis through the centre of mass, and I' the moment of inertia with reference to any parallel axis at a distance d, then if M is the mass of the body, we can easily prove the relation

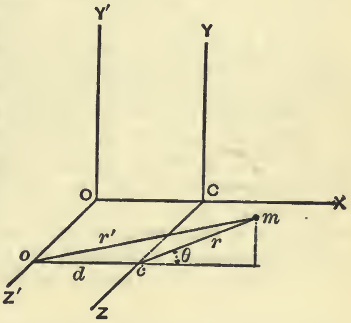
$$I' = I + Md^2.$$

That is, *the moment of inertia of a body with reference to any axis is equal to the moment of inertia with reference to a parallel axis through the centre of mass, plus the product of the mass of the body by the square of the distance between the two axes.*

This is called the *theorem of moment of inertia for parallel axes.* By means of it we can find I' for any axis, if I for a parallel axis through the centre of mass and the distance d between these parallel axes are given. Or conversely, we can find I if I' and d are known.

We can easily prove this theorem as follows: Let CZ be an axis through the centre of mass C, OZ' a parallel axis and d the perpendicular distance between these two axes.

Let m be the mass of any particle, r' its perpendicular distance mo from the axis OZ', r its perpendicular distance mc from the axis CZ, and θ the angle of r with $oc=d$ through the foot of these perpendiculars.



Then we have

$$I' = \Sigma mr'^2 \quad \text{and} \quad I = \Sigma mr^2.$$

But

$$r'^2 = r^2 + d^2 + 2rd \cos\theta.$$

Hence

$$\Sigma mr'^2 = \Sigma mr^2 + d^2\Sigma m + 2d\Sigma mr \cos\theta.$$

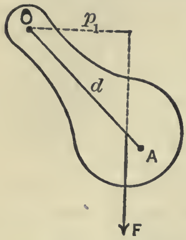
But $mr \cos\theta$ is the moment of m with reference to a plane YZ through the centre of mass. Hence

$$\Sigma mr \cos\theta = 0.$$

We have then

$$I' = I + Md^2.$$

Reduction of Mass.—Let a body rotate about a fixed axis through the centre of rotation O perpendicular to the plane of the paper with an angular velocity ω and an angular acceleration α, and let F be the resultant in the plane of rotation of all the external forces causing rotation, and p' its lever-arm.



Then (page 170) we have

$$Fp' = I'\alpha \quad \text{and} \quad \Sigma mvr' = I'\omega,$$

where I' is the moment of inertia of the body with reference to the axis at O, and $\Sigma mvr'$ is the moment of momentum of the body.

Let A be any point of the body at a distance d from the axis at O. Its linear acceleration is $f = d\alpha$ and its linear velocity is $v = d\omega$. Suppose the entire body *to be replaced by a single particle of mass M_1 at A.* Then the moment of the effective force of this particle $M_1 d^2\alpha$ must be equal to the sum of the moments $I'\alpha$ of the effective forces of all the particles of the body, and the moment of momentum of this particle $M_1 d^2\omega$ must be equal to the moment of momentum $I'\omega$ of all the particles of the body. Hence we must have

$$M_1 d^2\alpha = I'\alpha \quad \text{and} \quad M_1 d^2\omega = I'\omega,$$

or in both cases

$$M_1 = \frac{I'}{d^2}.$$

That is, when a body rotates about any axis, *we can reduce the body to an equivalent particle of mass M_1 at any distance d from the axis, by dividing the moment of inertia of the body I' with reference to this axis by the square of the distance.*

The moment of momentum and of the effective force of this single particle is the same as the moment of momentum and of the effective forces of all the particles of the body.

The mass of this particle is called the **reduced mass.**

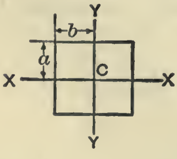
Moment of Inertia for a Rectangle, Ellipse or Ellipsoid about an Axis of Symmetry through the Centre of Mass.—We shall show hereafter how to find the moment of inertia for any body.

We give here without proof a simple rule which will enable the student to find at once the moment of inertia *with reference to an axis of symmetry through the centre of mass*, for a rectangle, ellipse or ellipsoid. This rule is as follows:

$$\left.\begin{matrix} I \\ \textit{about axis of} \\ \textit{symmetry} \\ \textit{through cen-} \\ \textit{tre of mass} \end{matrix}\right\} = \textbf{Mass} \times \frac{\textit{sum of squares of perpendicular semi-axes}}{3,\ 4 \text{ or } 5}.$$

The denominator 3, 4 or 5 is taken according as the body is rectangular, elliptical or ellipsoidal.

(1) *Rectangle and Right Parallelopipedon.*—Thus for a rectangular area whose sides are $2a$ and $2b$, about the axis of symmetry XX perpendicular to the side $2a$, we have, since a is the only perpendicular semi-axis,



$$I_x = \text{mass} \cdot \frac{a^2}{3}.$$

For the axis of symmetry YY perpendicular

– to $2b$ we have

$$I_y = \text{mass} \cdot \frac{b^2}{3}.$$

In both cases I is the same as for 1/3 *of the mass concentrated at a corner.*

For the axis through the centre of mass C perpendicular to the plane we have the two perpendicular semi-axes a and b, and hence

$$I_z = \text{mass} \cdot \frac{a^2 + b^2}{3},$$

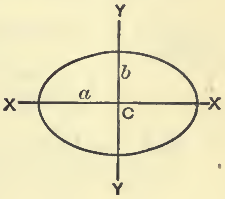
or 1/3 *of the mass concentrated at a corner.* The same holds for a right parallelopipedon, since it is composed of an indefinitely large number of indefinitely thin rectangles.

(2) *Ellipse.*—For an elliptical area let the semi-axes be a and b. Then for axis XX, or major axis a,

$$I_x = \text{mass} \cdot \frac{b^2}{4}.$$

For axis YY, or minor axis b,

$$I_y = \text{mass} \cdot \frac{a^2}{4}.$$



For axis through the centre of mass C perpendicular to the plane

$$I_z = \text{mass} \cdot \frac{a^2 + b^2}{4}.$$

This last holds for a right cylinder, since it is composed of an indefinitely large number of indefinitely thin ellipses.

For the special case of a *circular area,* $a = b = r$, and we have for any axis through the centre in the plane

$$I = \text{mass} \cdot \frac{r^2}{4},$$

and for axis through the centre at right angles to the plane

$$I_z = \text{mass} \cdot \frac{r^2}{2}.$$

This last holds for a right cylinder, since it is composed of an indefinitely large number of indefinitely thin circles.

(3) *Ellipsoid.*—For an ellipsoid whose semi-axes are a, b, c, we have for axis a

$$I_x = \text{mass} \cdot \frac{b^2 + c^2}{5}.$$

In the same way for axis b

$$I_y = \text{mass} \cdot \frac{a^2 + c^2}{5},$$

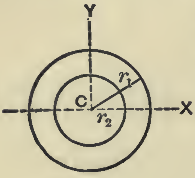
and for axis c

$$I_z = \text{mass} \cdot \frac{a^2 + b^2}{5}.$$

For the special case of a sphere, $a = b = c = r$, and for any axis through the centre

$$I = \text{mass} \, . \, \frac{2}{5} r^2.$$

Moment of Inertia for a Hollow Disk.—Let the outer radius be r_1 and the inner radius r_2.



Take the axis through the centre C at right angles to the plane of the disk. Let M_1 be the mass of a solid disk of radius r_1, and M_2 of a solid disk of radius r_2. Then from the preceding article we have for the moment of inertia of the hollow disk

$$I_z = M_1 \frac{r_1^2}{2} - M_2 \frac{r_2^2}{2}.$$

If δ is the surface density we have $M_1 = \delta \pi r_1^2$, $M_2 = \delta \pi r_2^2$. Hence

$$I_z = \frac{\delta\pi}{2}\left(r_1^4 - r_2^4\right) = \frac{\delta\pi}{2}\left(r_1^2 - r_2^2\right)\left(r_1^2 + r_2^2\right).$$

But $\delta\pi(r_1^2 - r_2^2) = M =$ mass of the hollow disk. Hence

$$I_z = \frac{M}{2}\left(r_1^2 + r_2^2\right).$$

This last holds for a hollow cylinder, since it is composed of an indefinitely large number of indefinitely thin disks.

We have then

$$I_x = I_y = \frac{M}{4}\left(r_1^2 + r_2^2\right).$$

Radius of Gyration.—We may conceive the mass of any body to be concentrated in a single point, so situated that the moment of inertia of this point with reference to any axis is the same as for the body itself with reference to the same axis. The distance of this point from the axis is called the **radius of gyration** for that axis.

The radius of gyration of a body for any axis, then, *is the distance from the axis to a point at which if the entire mass were concentrated its moment of inertia would be the same as that of the body itself with reference to the same axis.*

Let $\kappa' =$ the radius of gyration for any axis and κ the radius of gyration for a parallel axis through the centre of mass, M the mass of the body and d the distance between the parallel axes.

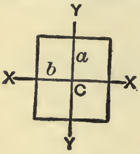
Then if I' is the moment of inertia for any axis and I for a parallel axis through the centre of mass at a distance d, we have $I' = M\kappa'^2$, $I = M\kappa^2$; and since $I' = I + Md^2$, or $I' = M\kappa'^2 = M\kappa^2 + Md^2$, we have

$$\kappa'^2 = \kappa^2 + d^2,$$

from which we see that κ' is a minimum for $d = 0$, in which case $\kappa' = \kappa$. That is, *the radius of gyration for an axis through the centre of mass is less than for any other parallel axis.*

We have also $\kappa'^2 = \dfrac{I'}{M}$, $\kappa^2 = \dfrac{I}{M}$; that is, *the square of the radius of gyration equals the moment of inertia divided by the mass.*

Thus from the results of page 174 we have for a rectilinear area or parallelopipedon with reference to axis XX



$$\kappa_x = \frac{a}{\sqrt{3}},$$

and with reference to axis YY

$$\kappa_y = \frac{b}{\sqrt{3}}.$$

With reference to axis through the centre of mass C perpendicular to the plane

$$\kappa_z = \frac{\sqrt{a^2 + b^2}}{\sqrt{3}}.$$

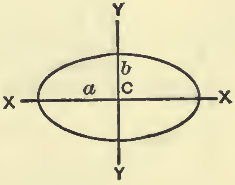
For an elliptical area with reference to axis XX

$$\kappa_x = \frac{b}{2},$$

with reference to axis YY

$$\kappa_y = \frac{a}{2},$$

with reference to axis through the centre of mass C perpendicular to the plane



$$\kappa_z = \frac{\sqrt{a^2 + b^2}}{2}.$$

This last holds for a right cylinder also.

For the special case of a *circular area*, for axis in plane through the centre

$$\kappa_x = \kappa_y = \frac{r}{2},$$

while for axis through centre perpendicular to the plane

$$\kappa_z = \frac{r}{\sqrt{2}}.$$

This last holds for a right cylinder also.

For an ellipsoid whose semi-axes are a, b and c we have,

for axis a,

$$\kappa_x = \frac{\sqrt{b^2 + c^2}}{\sqrt{5}},$$

for axis b,

$$\kappa_y = \frac{\sqrt{a^2 + c^2}}{\sqrt{5}},$$

for axis c,

$$\kappa_z = \frac{\sqrt{a^2 + b^2}}{\sqrt{5}}.$$

For the special case of a sphere

$$\kappa = r\sqrt{\frac{2}{5}}.$$

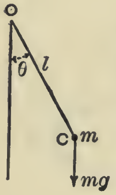
For a hollow disk

$$\kappa_x = \kappa_y = \frac{\sqrt{r_1^2 + r_2^2}}{2},$$

$$\kappa_z = \frac{\sqrt{r_1^2 + r_2^2}}{\sqrt{2}}.$$

This last holds for a right cylinder also.

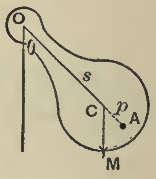
Compound Pendulum.—A material particle suspended from a fixed point by a string without mass and oscillating under the action of gravity is called a **simple pendulum.**



Let the mass of the particle be m, the point of suspension O, the length of the string $OC = l$, and the angle with the vertical θ. Then the moment of inertia of the particle m, with reference to O, is $ml^2 = I'$, the impressed force is $mg = F$, the moment of the impressed force is $-mg \times l \sin\theta$, and if α is the angular acceleration, we have (page 170)

$$-mg \times l \sin\theta = ml^2\alpha, \quad \text{or} \quad \alpha = -\frac{g \sin\theta}{l}. \quad . \quad (1)$$

A body of any form oscillating under the action of gravity about a fixed axis is called a **compound pendulum.** Let the axis of rotation be perpendicular to the plane of the paper. Let C be the centre of mass, also in the plane of the paper. Then the intersection O of the plane through C perpendicular to the axis with the axis is the **point or centre of suspension,** or **centre of rotation** (page 167).



Let $OC = s$ be the distance between the centre of mass C and the point of suspension O, and θ the angle of OC with the vertical. Then if M is the mass of the body, Mg is its weight acting at the centre of mass. The moment with reference to O of the impressed forces is then $-Mg \times s \sin\theta$. Let I' be the moment of inertia of the body with reference to the axis at O. Then if κ is the radius of gyration for a parallel line through the centre of mass, $I' = M(\kappa^2 + s^2)$ (page 176), and we have

$$-Mgs \sin\theta = M(\kappa^2 + s^2)\alpha, \quad \text{or} \quad \alpha = -\frac{gs \sin\theta}{\kappa^2 + s^2}. \quad . \quad . \quad (2)$$

If we equate (1) and (2), we find the length l of the *equivalent simple pendulum*, that is, the simple pendulum which has the same motion as the given compound pendulum,

$$l = \frac{\kappa^2 + s^2}{s} = \frac{I'}{Ms}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or, *the length of the equivalent simple pendulum is equal to the moment of inertia I′ with reference to the axis divided by the statical moment Ms.*

The time of vibration of the simple pendulum (Vol. I, *Kinema-*

tics, page 154) is $t = \pi\sqrt{\frac{l}{g}}$. Hence the time of vibration of the compound pendulum is

$$t = \pi\sqrt{\frac{\kappa^2 + s^2}{gs}} = \pi\sqrt{\frac{I'}{Mgs}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we prolong $OC = s$ to A and make

$$OA = l = \frac{\kappa^2 + s^2}{s} = \frac{I'}{Ms}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the point A thus obtained is called the **centre of oscillation**, because it is the point at which if the whole mass were concentrated the motion would be the same as for a simple pendulum.

We have then

$$CA = p = l - s,$$

or

$$CA = p = \frac{\kappa^2}{s} = \frac{I'}{Ms}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is,

the centre of oscillation is the same as the centre of percussion (page 181), and

the radius of gyration κ *is a mean proportional between OC and CA.*

Also, *the distance CA is equal to the moment of inertia I' divided by the statical moment Ms, and the distance OA is equal to the moment of inertia I' divided by the statical moment Ms.*

Now suppose the point of suspension is at A instead of O.

Then from (3) the length of the equivalent simple pendulum will be

$$l = \frac{\kappa^2 + p^2}{p}.$$

If we insert in this the value of p from (6), we have

$$l = \frac{\kappa^2 + s^2}{s} = \frac{I'}{Ms},$$

which is just the same as equation (3).

Hence, *the centre of suspension and oscillation can be interchanged without changing the time of oscillation.*

Experimental Determination of Moment of Inertia.—From these principles we can determine experimentally the moment of inertia of a body with reference to any axis.

1st. Thus first determine the mass M of a body by weighing it. Then suspend it from an axis and note the time t of vibration. The length of the equivalent simple pendulum is then

$$l = \frac{gt^2}{\pi^2}.$$

Now balance the body on a knife-edge parallel to the axis of suspension and thus find the distance $s = OC$ from the point of suspension O to to the centre of mass C. Then the moment of inertia with reference to the axis of suspension is

$$I' = Msl.$$

We have also $p = l - s$ and $\kappa^2 = ps = ls - s^2$. Hence the moment of inertia with reference to a line through the centre of mass parallel to the axis of suspension is

$$I = M\kappa^2.$$

2d. First determine the mass M of the body by weighing it. Then suspend it from an axis and note the time t of vibration. Then turn it over and find by trial another parallel axis from which, when, it is suspended, the time of vibration is the same. The distance between these axes is the length l of the equivalent simple pendulum. Then balance the body on a knife-edge and thus find the distances s and p of the centre of mass from the two axes. Then we have $\kappa^2 = sp$ and

$$I = M\kappa^2.$$

Also if we have thus measured l and t, we have the value of g at the place of observation

$$g = \frac{\pi^2 l}{t^2}.$$

We may also determine the moment of inertia experimentally as follows:

3d. First determine the mass M of the body by weighing it. Then suspend it from an axis and note the time t of vibration. Then attach a spring-balance to the lower end and raise the lower end until the centre of mass is in a horizontal through the axis, and note the reading F of the balance. This position is reached when the reading F is a maximum.

If L is the distance in feet from the axis to the point of attachment of the balance, and F is the reading of the balance in pounds, we have

$$Ms = FL.$$

Hence, since

$$I' = Msl \quad \text{and} \quad l = \frac{gt^2}{\pi^2},$$

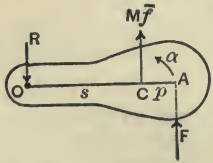
we have

$$I' = \frac{FLgt^2}{\pi^2}.$$

For the parallel axis through the centre of mass

$$I = I' - Ms^2.$$

Centre of Percussion.—Let C be the centre of mass of a body of mass M rotating about a fixed axis, and O the centre of rotation, so that the distance $OC = s$. Suppose the body is struck so that the force of impact F is in the plane of rotation and at right angles to OC, intersecting OC at A. Let $CA = p$.



Then if κ is the radius of gyration of the body with reference to the line through C parallel to the axis of rotation through O, we have for the moment of inertia for the axis through O (page 176) $I' = M(\kappa^2 + s^2)$. The moment of the impressed force is $F(s + p)$, and (page 170)

$$I'\alpha = F(s + p), \quad \text{or} \quad M(\kappa^2 + s^2)\alpha = F(s + p);$$

hence

$$\alpha = \frac{F(s + p)}{M(\kappa^2 + s^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where α is the angular acceleration about the axis at O.

Now the pressure on the fixed axis is R parallel to F. Since the centre of mass moves as if all the mass and all the impressed forces were collected at the centre of mass, if $\bar{f}$ is the acceleration of the centre of mass, we have

$$M\bar{f} = F + R.$$

But $\bar{f} = s\alpha$. Hence

$$Ms\alpha = F + R, \quad \text{or} \quad \alpha = \frac{F + R}{Ms}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating (1) and (2), we obtain for the reaction of the axis at O

$$R = \frac{F(ps - \kappa^2)}{\kappa^2 + s^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The centre of mass has then the linear acceleration

$$\bar{f} = \frac{F + R}{M},$$

while at the same time the body rotates about the centre of mass with the angular acceleration α.

If $ps > \kappa^2$ we see from (3) that R is positive or in the same direction as F.

If $ps < \kappa^2$ we have R negative or opposite in direction to F.

If $ps = \kappa^2$, or

$$CA = p = \frac{\kappa^2}{s} = \frac{I}{Ms}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the reaction R of the axis at O is zero. In this latter case we have then

$$OA = p + s = \frac{\kappa^2 + s^2}{s} = \frac{I'}{Ms}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The point A given by (5) is called the **centre of percussion**, because if a body is struck at this point so that the impulse is in the plane of rotation and at right angles to OC, there *will be no reaction of the axis.*

Hence, *the centre of percussion is the same as the centre of oscillation* (page 179), and

the radius of gyration κ is a mean proportional between OC and CA.

Also, *the distance CA is equal to the moment of inertia I divided by the statical moment Ms, and the distance OA is equal to the moment of inertia I′ divided by the statical moment Ms.*

We may obtain the same result as follows :

The mass of the body *reduced to the point A* (page 174) is

$$\frac{I'}{(p + s)^2}.$$

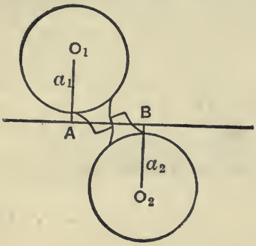
The acceleration of the point A is $(p + s)\alpha$. The force on the reduced mass is then

$$\frac{I'\alpha}{p + s},$$

and this force must equal the algebraic sum of all the impressed forces $F + R = M\bar{f} = Ms\alpha$. Hence

$$\frac{I'\alpha}{p+s} = Ms\alpha, \quad \text{or} \quad \frac{I'}{Ms} = p + s.$$

Impact of Revolving Bodies.—Let two bodies of mass m_1 and m_2 revolve about fixed axes at O_1 and O_2 and impinge, and let AB be the line of impact. Let the normals $O_1A = a_1$ and $O_2B = a_2$.



Let κ_1' and κ_2' be the radii of gyration of the bodies with reference to the axes at O_1 and O_2. Then (page 174) we can reduce the masses m_1 and m_2 to the equivalent masses

$$\frac{m_1\kappa_1'^2}{a_1^2} \quad \text{and} \quad \frac{m_2\kappa_2'^2}{a_2^2}$$

at A and B. If then we substitute these masses in the place of m_1 and m_2 in the equations for central impact (page 149) we have for bodies of the same material

$$\left.\begin{aligned} v_1 &= u_1 - (u_1 - u_2)(1+e)\,\frac{m_2\kappa_2'^2 a_1^2}{m_1\kappa_1'^2 a_2^2 + m_2\kappa_2'^2 a_1^2}, \\ v_2 &= u_2 + (u_1 - u_2)(1+e)\,\frac{m_1\kappa_1'^2 a_2^2}{m_1\kappa_1'^2 a_2^2 + m_2\kappa_2'^2 a_1^2}, \end{aligned}\right\} \quad . \quad . \quad (1)$$

where u_1 and v_1 are the velocities of A and B before and v_1, v_2 after impact, and e is the modulus of elasticity.

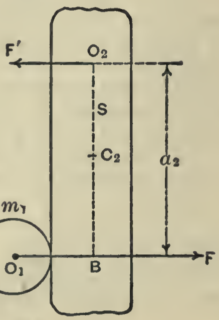
If ϵ_1 and ϵ_2 are the angular velocities before and ω_1, ω_2 the angular velocities after impact, we have, taking counter-clockwise rotation as positive, the origins at O_1 and O_2, and a_1, a_2 as coinciding with the axes of Y for each origin,

$$u_1 = -a_1\epsilon_1, \quad u_2 = -a_2\epsilon_2, \quad v_1 = -a_1\omega_1, \quad v_2 = -a_2\omega_2$$

Hence

$$\left.\begin{aligned} \omega_1 &= \epsilon_1 - a_1(a_1\epsilon_1 - a_2\epsilon_2)(1+e)\,\frac{m_2\kappa_2'^2}{m_1\kappa_1'^2 a_2^2 + m_2\kappa_2'^2 a_1^2}, \\ \omega_2 &= \epsilon_2 + a_2(a_1\epsilon_1 - a_2\epsilon_2)(1+e)\,\frac{m_1\kappa_1'^2}{m_1\kappa_1'^2 a_2^2 + m_2\kappa_2'^2 a_1^2}, \end{aligned}\right\} \quad . \quad (2)$$

Impact of an Oscillating Body.—If the body of mass m_1 has a motion of translation only and impinges upon m_2, which is suspended from an axis at O_2, the equations of the preceding article apply if we put in equations (1) m_2 in place of $\frac{m_1\kappa_1'^2}{a_1^2}$, and $-a_1\epsilon_1$ in place of u_2 and $-a_2\omega_2$ in place of v_1. We have then for the velocity of the mass m_1 after impact, taking counter-clockwise rotation as positive, and O_2C_2 as coinciding with the axis of Y and origin at O_2,



$$v_1 = u_1 - (u_1 + a_2\epsilon_2)(1+e)\,\frac{m_2\kappa_2'^2}{m_1 a_2^2 + m_2\kappa_2'^2}, \quad . \quad . \quad . \quad (1)$$

and for the angular velocity of the mass m_2 after impact

$$\omega_2 = \epsilon_2 - a_2(u_1 + a_2\epsilon_2)(1 + e)\frac{m_1}{m_1a_2^2 + m_2\kappa_2'^2}. \quad . \quad . \quad (2)$$

If the mass m_2 were at rest before impact we have $\epsilon_2 = 0$, and

$$v_1 = u_1 - u_1(1 + e)\frac{m_2\kappa_2'^2}{m_1a_2^2 + m_2\kappa_2'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\omega_2 = - u_1(1 + e)\frac{m_1a_2}{m_1a_2^2 + m_2\kappa_2'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If m_1 were at rest and the oscillating mass m_1 impinges on it, we have $u_1 = 0$, and hence

$$v_1 = - \epsilon_2(1 + e)\frac{m_2\kappa_2'^2a_2}{m_1a_2^2 + m_2\kappa_2'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\omega_2 = \epsilon_2\left[1 - (1 + e)\frac{m_1a_2^2}{m_1a_2^2 + m_2\kappa_2'^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The velocity ω_2 of m_2 in the first case, equations (4), or the velocity v_1 of m_1 in the second case, equation (5), is a maximum when

$$\frac{a_2}{m_1a_2^2 + m_1\kappa_2'^2}$$

is a maximum, or when

$$m_1a_2 + \frac{m_2\kappa_2'^2}{a_2}$$

is a minimum. Putting the first differential coefficient equal to zero, we have for the value of a_2 when the maximum velocity is imparted

$$a_2 = \kappa_2'\sqrt{\frac{m_2}{m_1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence the maximum velocity imparted to the oscillating body m_2 when at rest and struck by m_1 is given by

$$\omega_2 = - (1 + e)\frac{u_1}{2\kappa_2'}\sqrt{\frac{m_1}{m_2}} = - (1 + e)\frac{u_1}{2a_2}, \quad . \quad . \quad . \quad . \quad (8)$$

and the maximum velocity imparted to the free body m_1 when at rest and struck by the oscillating body is

$$v_1 = - \frac{1}{2}\kappa'\epsilon_2(1 + e)\sqrt{\frac{m_2}{m_1}} = - (1 + e)\frac{\epsilon_2a_2}{2}. \quad . \quad . \quad . \quad (9)$$

Reaction of the Axis.—Let the force of impact be F, and the reaction of the axis be F'. Then, from page 181, we have

$$F' = F\left(\frac{a_2s}{\kappa_2'^2} - 1\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If we give to a_2 its value from (7), we have for the reaction of the axis when the maximum velocity is imparted

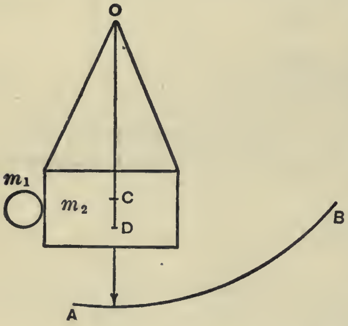
$$F' = F\left(\frac{s}{\kappa_2'}\sqrt{\frac{m_2}{m_1}} - 1\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The centre of percussion (page 181) is at the distance

$$a_2 = \frac{\kappa_2'^2}{s} = \frac{I'}{Ms}$$

from the axis. If the impact takes place at this distance there is no reaction of the axis.

Ballistic Pendulum.—The ballistic pendulum consists of a large mass m_2 which is hung from a horizontal axis O. It is set in oscillation by a cannon-ball shot against it, and is used to determine the velocity of the ball. In order to render the impact inelastic the mass m_2 consists of a box filled with sand or clay, so that the ball enters the mass and oscillates with it.



In order to determine the velocity of the ball the angle of oscillation is measured by a pointer directly below the centre of mass which moves on a graduated arc AB.

Let m_1 be the mass of the ball. Then from equation (4) of the preceding article, making $e = 0$, we have for the angular velocity after impact

$$\omega = -\frac{m_1 a_2 u_1}{m_1 a_2^2 + m_2 \kappa_2'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $m_2 \kappa_2'^2$ is the moment of inertia of the pendulum with reference to the axis O, or κ_2' is its radius of gyration with reference to this axis, and a_2 is the distance of the point of impact below the axis.

Let l be the length of the equivalent simple pendulum which oscillates in the same time as the ballistic pendulum, and let the angle of displacement be θ, as measured on the arc AB.

We have then for the simple pendulum (page 178) the angular acceleration

$$\alpha = \frac{g \sin \theta}{l}.$$

If $OC = s$ is the distance of the centre of mass of the compound pendulum from the axis, we have (page 170)

$$(m_1 + m_2) gs \sin \theta = I'\alpha = (m_1 a_2^2 + m_2 \kappa_2'^2)\alpha,$$

or

$$\alpha = \frac{(m_1 + m_2) gs \sin \theta}{m_1 a_2^2 + m_2 \kappa_2'^2}.$$

Equating these two values of α, we obtain

$$l = \frac{m_1 a_2^2 + m_2 \kappa_2'^2}{(m_1 + m_2)s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The height of displacement is

$$h = l - l \cos \theta = 2l \sin^2 \frac{\theta}{2}.$$

Hence the velocity at the lowest point of the path is

$$v = \sqrt{2gh} = 2 \sqrt{gl} . \sin \frac{\theta}{2},$$

and the corresponding angular velocity is

$$\omega = - \frac{v}{l} = - 2 \sqrt{\frac{g}{l}} . \sin \frac{\theta}{2}.$$

Equating this to (1) and inserting the value of l from (2), we obtain for the velocity of the ball

$$u_1 = 2\left(\frac{m_1 + m_2}{m_1}\right) \frac{s}{a_2} \sqrt{gl} . \sin \frac{\theta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

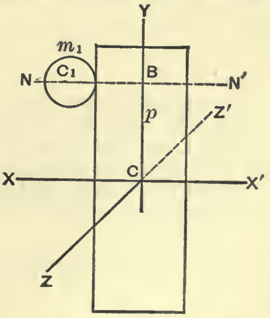
If the pendulum makes n vibrations per minute, the duration of a vibration is

$$t = \pi \sqrt{\frac{l}{g}} = \frac{60}{n}, \quad \text{and therefore} \quad \sqrt{gl} = \frac{60g}{n\pi}.$$

Hence the required velocity of the ball is

$$u_1 = \frac{m_1 + m_2}{m_1} . \frac{120gs}{n\pi a_2} . \sin \frac{\theta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eccentric Impact.—Let the two masses m_1 and m_2 be perfectly free. Let the mass m_1 strike the mass m_2 in such a manner that the direction NN' of the line of impact passes through the centre of mass C_1 of m_1, but not through the centre of mass C of m_2. The impact is then central for m_1 and eccentric for m_2.



Let m_1 have a motion of translation only, and let its initial and final velocities be u_1 and v_1 in the line of impact NN'. Through the centre of mass C of m_2 take the origin at C and the axis of X parallel to NN', and let the direction of u_1 be positive. Take the axis of Y through C at right angles to NN', and let B be its intersection with NN'. Denote the distance CB by p, positive when above and negative when below the axis of X. Let the initial and final velocities of translation of m_2 parallel to NN' be u_2 and v_2, and its initial and final angular velocities about the axis ZZ through C at right angles to the plane XY be ϵ_2 and ω_2, and let counter-clockwise rotation be positive.

Let κ_z be the radius of gyration of m_2 with reference to the axis ZZ. Then the moment of inertia with reference to this axis is $I = m_2\kappa_z^2$.

Now from equation (I), page 146, we have for the impact, so far as translation is concerned,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The mass of m_2 reduced to the point B is (page 174)

$$\frac{m_2\kappa_z^2}{p^2},$$

and the linear velocities of the point B before and after impact, *due to rotation*, are $-p\epsilon_2$ and $-p\omega_2$, or opposite in direction to u_1 when p, ϵ_2 and ω_2 are positive.

We have then for impact, so far as rotation is concerned,

$$m_1u_1 - \frac{m_2\kappa_z^2}{p^2} \cdot p\epsilon_2 = m_1v_1 - \frac{m_2\kappa_z^2}{p^2} \cdot p\omega_2. \quad . \quad . \quad . \quad (2)$$

If the bodies *are inelastic*, m_1 and the point B move together after impact, and we have

$$v_1 = v_2 - p\omega_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating v_2 and ω_2 from (1) and (2) by means of (3), we obtain for the loss of velocity of m_1

$$u_1 - v_1 = \frac{m_1\kappa_z^2(u_1 - u_2 + p\epsilon_2)}{(m^2 + m_2)\kappa_z^2 + m_1p^2};$$

for the gain of velocity of translation of m_2

$$v_2 - u_2 = \frac{m_1\kappa_z^2(u_1 - u_2 + p\epsilon_2)}{(m_1 + m_2)\kappa_z^2 + m_1p^2};$$

and for the gain of angular velocity of m_2

$$\omega_2 - \epsilon_2 = -\frac{m_1p(u_1 - u_2 + p\epsilon_2)}{(m_1 + m_2)\kappa_z^2 + m_1p^2}.$$

If the bodies are perfectly elastic these values are twice as great (page 151). If the bodies are imperfectly elastic and of the same material, so that e is the common modulus of elasticity, these values are $(1 + e)$ times as great (page 149).

We have then in general for imperfectly elastic bodies of the same material

$$v_1 = u_1 - \left(u_1 - u_2 + p\epsilon_2\right)\frac{(1 + e)m_2\kappa_z^2}{(m_1 + m_2)\kappa_z^2 + m_1p^2}; \quad . \quad (4)$$

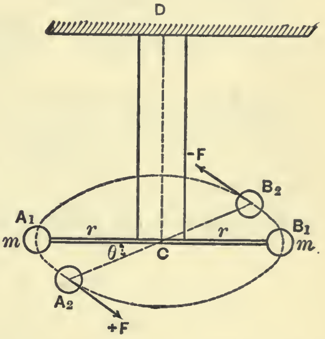
$$v_2 = u + \left(u_1 - u_2 + p\epsilon_2\right)\frac{(1 + e)m_1\kappa_z^2}{(m_1 + m_2)\kappa_z^2 + m_1p^2}; \quad . \quad . \quad . \quad (5)$$

$$\omega_2 = \epsilon_2 - \left(u_1 - u_2 + p\epsilon_2\right)\frac{(1 + e)m_1p}{(m_1 + m_2)\kappa_z^2 + m_1p^2}. \quad . \quad . \quad . \quad (6)$$

These equations are general. If the impact is central, $p = 0$, and ω_2 in (6) is unchanged and equal to ϵ_2, while (4) and (5) reduce to equations (VII), page 149.

If the bodies are perfectly elastic, $e = 1$. If non-elastic, $e = 0$. If m_2 moves towards m_1, u is negative. If m_1 is initially at rest, $u_1 = 0$. If m_2 is initially at rest, $u_2 = 0$ and $\epsilon_2 = 0$. If m_1 is fixed we can take $m_1 = \infty = m_1 + m_2$, and $u_1 = 0$.

[**Torsion-pendulum.**—Let an elastic bar CD of uniform cross-section be fixed at one end D, and at the other end C have two equal masses, m, m, rigidly fixed to it by a cross-bar A_1B_1 of uniform cross-section. Let the cross-bar be of equal arms, so that the distance $A_1C = B_1C = r$, A_1 and B_1 being the centres of mass of the masses m, m, and C the centre of mass of the end cross-section of the bar CD.



If now the cross-bar be turned into the position A_2B_2, making the angle $A_1CA_2 = \theta$ with its original position, and then released, it will vibrate back and forth. Such an arrangement is called a *torsion-pendulum.*

1*st.* *To find the force necessary to twist the bar CD through a given angle.*

Let $\theta = A_1CA_2$ be the angle of twist, and A_1B_1 be the neutral position of the cross-bar.

Let the forces $+F$, $-F$ act upon the cross-bar at A_2 and B_2, at right angles to the cross-bar. Then, *if the limit of elasticity of CD is not exceeded,* we have (Vol. II, *Statics*, page 310) for equilibrium

$$2Fr = \frac{EI_z}{l}\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where E is the coefficient of elasticity for the material of the bar CD, l is the length of the bar CD, θ is the angle of twist A_1CA_2, and I_z is the *polar* moment of inertia *of the cross-section* of the bar CD (Vol. II, *Statics*, page 271).

We have then

$$F = \frac{EI_z}{2rl}\theta;$$

or if the angle θ is measured from some fixed line with which the neutral position of A_1B_1 makes the angle ϕ, the angle of twist is $\theta - \phi$ and

$$F = \frac{EI_z}{2rl}(\theta - \phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

2*d.* *To find the time of an oscillation.*

Now let the forces $+F$, $-F$ be removed. The cross-bar will vibrate back and forth.

Let I be the moment of inertia of the cross-bar and masses m, m with reference to CD as an axis.

The angular acceleration at any instant for which the angle of twist is β is

$$\alpha = \frac{d^2\beta}{dt^2},$$

and the moment of the elastic forces is, from (1),

$$\frac{EI_z}{l}\beta.$$

We have then (page 170)

$$I\frac{d^2\beta}{dt^2} = -\frac{EI_z}{l}\beta.$$

Multiplying by $d\beta$ and integrating, we have

$$\frac{d\beta^2}{dt^2} = -\frac{I_zE}{Il}\beta^2 + \text{Const.}$$

When $\beta = \theta$, the angular velocity $\omega = \frac{d\beta}{dt}$ is zero. Hence

$$\text{Const.} = \frac{I_zE}{Il}\theta^2$$

and

$$\frac{d\beta^2}{dt^2} = \frac{I_zE}{Il}(\theta^2 - \beta^2),$$

or

$$dt = \sqrt{\frac{Il}{I_zE}}\,\frac{d\beta}{\sqrt{\theta^2 - \beta^2}}.$$

Integrating again, we obtain, if $t = 0$ when $\beta = 0$,

$$t = \sqrt{\frac{Il}{I_zE}}\,\sin^{-1}\frac{\beta}{\theta},$$

which between the limits of $\beta = +\theta$ and $\beta = -\theta$ gives for the time of an oscillation

$$T = \pi\sqrt{\frac{Il}{I_zE}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we substitute the value of E from (2), we have

$$T = \pi\sqrt{\frac{I(\theta - \phi)}{2dF}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

from which we obtain

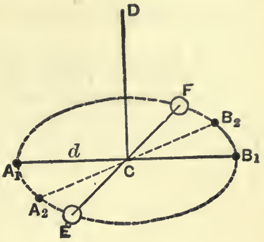
$$2rF = I(\theta - \phi)\frac{\pi^2}{T^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Equation (5) gives the moment $2rF$ if the time of oscillation T and the angle of twist $(\theta - \phi)$ are observed. The force F is of course in poundals.

All these formulas of course hold good only *provided the limit of elasticity of CD is not exceeded.*

3*d*. *To find the density of the earth by the torsion-pendulum.*

The plan of determining the density of the earth by means of the torsion-pendulum was first suggested by the Rev. John Mitchel and executed by Mr. Cavendish in 1798. At the request of the Astronomical Society of England Mr. Bailey made a new determination and published the results of upwards of 2000 experiments with balls of different weights and sizes suspended in a variety of ways, in the *Memoirs of the Astronomical Society*, Vol. XIV.



The torsion-rod was very slight, so that it could be easily twisted. Two small balls A_1, B_1 of mass m, m were suspended from the torsion-rod by a light cross-bar. Two large balls E, F of mass M, M were placed on a platform which turned about a pivot directly under C, so arranged that the centres of mass of the four balls were in the same horizontal plane.

The masses at E, F were first placed at right angles to the cross-bar in its neutral position A_1B_1, which was noted. Then the masses at E, F were brought quite near to the small masses at A_1, B_1, so that their mutual attraction caused an oscillation back and forth on each side of the neutral position, and the time of oscillation T and angle $\theta - \phi = A_1CA_2$ noted.

We have then from (5)

$$2rF = I(\theta - \phi)\frac{\pi^2}{T^2}.$$

But (Vol. II, *Statics*, page 48) the force of attraction between the masses m and M is

$$F = \frac{gR^2}{E} \cdot \frac{mM}{d^2},$$

where R is the radius of the earth and E its mass, and d the distance A_2E between the centres of mass of the balls at E and A_2.

Substituting this value of F, we obtain for the mass of the earth

$$E = \frac{2grR^2T^2mM}{I(\theta - \phi)\pi^2d^2}.$$

If a is the radius of the small balls of mass m at A_1 and B_1, and b is the mass of the cross-bar, the moment of inertia

$$I = 2\left[m\left(\frac{2}{5}a^2 + r^2\right) + \frac{b}{6}r^2\right],$$

and we have

$$E = \frac{grR^2T^2M}{\left(\frac{2}{5}a^2 + r^2 + \frac{b}{6m}r^2\right)(\theta - \phi)\pi^2d^2}.$$

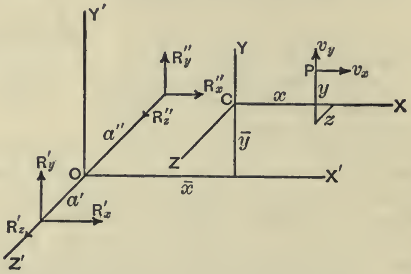
If ϵ is the specific mass of the earth and γ is the mass of a unit volume of water, we have $E = \frac{4}{3}\pi R^3 . \epsilon\gamma$, and hence

$$\epsilon = \frac{3grT^2M}{4\gamma R\left(\frac{2}{5}a^2 + r^2 + \frac{b}{6m}r^2\right)(\theta - \phi)\pi^2d^2}.$$

The experiments of Bailey gave $\epsilon = 5.6747$, or the mean density of the earth is 5.6747 times that of distilled water at its maximum density.

General Formulas for Rotation about a Fixed Axis.—Let a rigid body of mass M rotate about a fixed axis OZ' with the angular velocity ω_z and the angular acceleration $\alpha_z = \dfrac{d\omega}{dt}$.

Let C be the centre of mass. Pass a plane through C perpendicular to the fixed axis. Then this plane is the plane of rotation and its intersection O with the fixed axis is the centre of rotation (page 167).



Take O as the origin, the fixed axis as the co-ordinate axis of Z', and the other two co-ordinate axes X', Y' in the plane of rotation.

Let positive rotation be counter-clockwise. Through the centre of mass C take the co-ordinate axes CX, CY, CZ parallel at any instant to OX', OY', OZ'.

Let the co-ordinates of C with reference to O be $\bar{x}$, $\bar{y}$, and the co-ordinates of any point P of the body in general with reference to O be x', y', z', and with reference to C be x, y, z.

In the same way let the components of the velocity of C with reference to O be $\overline{v_x}$, $\overline{v_y}$, $\overline{v_z} = 0$, and the components of the velocity of any point P of the body in general with reference to O be v_x', v_y', $v_z' = 0$, and with reference to C be v_x, v_y, $v_z = 0$.

So also let the components of the acceleration of C with reference to O be $\overline{f_x}$, $\overline{f_y}$, $\overline{f_z} = 0$, and the components of the acceleration of any point P of the body in general with reference to O be f_x', f_y', $f_z' = 0$, and with reference to C be f_x, f_y, $f_z = 0$.

Then we have by reason of our rotation

$$x' = \bar{x} + x, \quad y' = \bar{y} + y, \quad z' = z.$$

For the components at any instant of the velocity of any point P of the body, due to rotation about the fixed axis OZ', we have then

$$\left.\begin{aligned} \frac{dx'}{dt} &= v_x' = -y'\omega_z = -\bar{y}\omega_z - y\omega_z; \\ \frac{dy'}{dt} &= v_y' = x'\omega_z = \bar{x}\omega_z + x\omega_z; \\ \frac{dz'}{dt} &= v_z' = 0. \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

If in (1) we make $\bar{x}$ and $\bar{y}$ zero we have the components v_x, v_y, v_z for rotation about CZ. If we make x and y zero we have the components

$\overline{v_x}$, $\overline{v_y}$, $\overline{v_z}$ of the velocity of the centre of mass C due to rotation about OZ'. Hence

$$\left.\begin{aligned}&\frac{dx}{dt} = v_x = -y\omega_z, \quad \frac{dy}{dt} = v_y = x\omega_z, \quad \frac{dz}{dt} = v_z = 0;\\&\frac{d\bar{x}}{dt} = \overline{v_x} = -\bar{y}\omega_z, \quad \frac{d\bar{y}}{dt} = \overline{v_y} = \bar{x}\omega_z, \quad \frac{d\bar{z}}{dt} = \overline{v_z} = 0.\end{aligned}\right\} . \quad (1a)$$

The components at any instant of the tangential acceleration of any point P of the body due to rotation about the fixed axis OZ' are in like manner with (1)

$$\begin{aligned}&f'_{tx} = -y'\alpha_z = -\bar{y}\alpha_z - y\alpha_z;\\&f'_{ty} = x'\alpha_z = \bar{x}\alpha_z + x\alpha_z;\\&f'_{tz} = 0.\end{aligned}$$

The components at any instant of the normal acceleration of any point P of the body due to rotation about the fixed axis OZ' are

$$\begin{aligned}&f'_{nx} = -x'\omega_z^2 = -\bar{x}\omega_z^2 - x\omega_z^2;\\&f'_{ny} = -y'\omega_z^2 = -\bar{y}\omega_z^2 - y\omega_z^2;\\&f'_{nz} = 0.\end{aligned}$$

Hence the components at any instant of the total acceleration of any point P due to rotation about the fixed axis OZ' are

$$\left.\begin{aligned}\frac{d^2x'}{dt^2} &= f'_x = f'_{tx} + f'_{nx} = -y'\alpha_z - x'\omega_z^2\\&= -\bar{y}\alpha_z - y\alpha_z - \bar{x}\omega_z^2 - x\omega_z^2;\\\frac{d^2y'}{dt^2} &= f'_y = f'_{ty} + f'_{ny} = x'\alpha_z - y'\omega_z^2\\&= \bar{x}\alpha_z + x\alpha_z - \bar{y}\omega_z^2 - y\omega_z^2;\\\frac{d^2z'}{dt^2} &= f'_z = f'_{tz} + f'_{nz} = 0.\end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

If in (2) we make $\bar{x}$ and $\bar{y}$ zero we have the components f_x, f_y, f_z for rotation about CZ. If we make x, y zero, we have the components $\overline{f_x}$, $\overline{f_y}$, $\overline{f_z}$ of the centre of mass C due to rotation about OZ'. Hence

$$\left.\begin{aligned}&\frac{d^2x}{dt^2} = f_x = -y\alpha_z - x\omega_z^2, \quad \frac{d^2y}{dt^2} = f_y = x\alpha_z - y\omega_z^2, \quad \frac{d^2z}{dt^2} = f_z = 0;\\&\frac{d^2\bar{x}}{dt^2} = \overline{f_x} = -\bar{y}\alpha_z - \bar{x}\omega_z^2, \quad \frac{d^2\bar{y}}{dt^2} = \overline{f_y} = \bar{x}\alpha_z - \bar{y}\omega_z^2, \quad \frac{d^2\bar{z}}{dt^2} = \overline{f_z}\, 0. =\end{aligned}\right\} . \quad (2a)$$

We can obtain (2) and (2a) directly from (1) and (1a) by differentiating, since $\frac{d\omega_z}{dt} = \alpha_z$.

Since CZ passes through the centre of mass, we have

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, if m is the mass of a particle,

$$\Sigma m = M, \quad \ldots \ldots \quad (4)$$

and

$$\left.\begin{array}{l} \Sigma m(x^2 = y^2) \ \ = I_z \times \text{moment of inertia for axis } CZ; \\ \Sigma m(x'^2 = y'^2) = I_z' \times \text{`` `` `` `` ``} \ OZ'. \end{array}\right\} \ldots (5)$$

Motion of Centre of Mass.—From (2) and (2a) we have for the sum of the components of all the *effective* forces (page 168) after reduction by (3) and (4)

$$\left.\begin{array}{l} \Sigma m f_x' = -M\bar{y}\alpha_z - M\bar{x}\omega_z^2 = M\bar{f}_x; \\ \Sigma m f_y' = M\bar{x}\alpha_z - M\bar{y}\omega_z = M\bar{f}_y; \\ \Sigma m f_z = 0. \end{array}\right\} \ldots \ldots (6)$$

But by D'Alembert's principle (page 168) the sum of the components of the impressed forces in any direction is equal to the sum of the components of the effective forces in that direction.

Hence, *the centre of mass moves at any instant as if all the mass and impressed forces were collected at the centre of mass.*

Momentum.—From (1) and (1a) we have for the sum of the components of momentum of all the particles, after reduction by (3) and (4),

$$\left.\begin{array}{l} \Sigma m v_x' = -M\bar{y}\omega_z = M\bar{v}_x; \\ \Sigma m v_y' = M\bar{x}\omega_z = M\bar{v}_y; \\ \Sigma m v_z' = 0. \end{array}\right\} \ldots \ldots (7)$$

Hence, *the momentum of the body is the same as for all the mass collected at the centre of mass.*

Moment of Momentum.—Let $\mathfrak{M}'_{mx}$, $\mathfrak{M}'_{my}$, $\mathfrak{M}'_{mz}$ be the sum of the moments of momentum of all the particles about the co-ordinate axes OX', OY', OZ' for any fixed axis of rotation OZ', and $\mathfrak{M}_{mx}$, $\mathfrak{M}_{my}$, $\mathfrak{M}_{mz}$ for the co-ordinate axes CX, CY, CZ. Then we have from (1), after reduction by (3) and (5),

$$\left.\begin{array}{l} \mathfrak{M}'_{mx} = \Sigma m(v_z'y' - v_y'z') = -\omega_z \Sigma m x'z' = -\omega_z \Sigma m xz = \mathfrak{M}_{mx}; \\ \mathfrak{M}'_{my} = \Sigma m(v_x'z' - v_z'x') = -\omega_z \Sigma m y'z' = -\omega_z \Sigma m yz = \mathfrak{M}_{my}; \\ \mathfrak{M}'_{mz} = \Sigma m(v_y'x' - v_x'y') = I_z'\omega_z. \end{array}\right\} \quad (8)$$

The last of these equations is equation (II), page 171.

We shall see in the next chapter that at any point of a body there are at least three rectangular axes for which we have

$$\Sigma m x'y' = 0, \quad \Sigma m x'z' = 0, \quad \Sigma m y'z' = 0.$$

These axes are called **principal axes** for that point. Hence if the fixed axis OZ' is a principal axis, we have, taking the other two principal axes as co-ordinate axes,

$$\mathfrak{M}'_{mx} = 0, \quad \mathfrak{M}'_{my} = 0, \quad \mathfrak{M}'_{mz} = I_z'\omega_z.$$

The resultant moment of momentum is in general

$$\mathfrak{M}'_m = \sqrt{\mathfrak{M}'^2_{mx} + \mathfrak{M}'^2_{my} + \mathfrak{M}'^2_{mz}}, \quad \ldots \ldots (9)$$

and the direction-cosines of the resultant axis of moment of momentum are then

$$\frac{\mathfrak{M}'_{mx}}{\mathfrak{M}'_m},\quad \frac{\mathfrak{M}'_{my}}{\mathfrak{M}'_m},\quad \frac{\mathfrak{M}'_{mz}}{\mathfrak{M}'_m}. \qquad (10)$$

If the fixed axis OZ' then *is a principal axis, the resultant axis of moment of momentum coincides with the fixed axis.*

Kinetic Energy.—Let v' be the velocity of any particle and $\bar{v}$ the velocity of the centre of mass C with reference to O (figure, page 190), so that

$$v'^2 = v_x'^2 + v_y'^2, \qquad \bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2.$$

Then we have, from (1),

$$\frac{1}{2}mv_x'^2 = \frac{1}{2}m\bar{y}^2\omega_z^2 - m\bar{y}\,y\omega_z^2 + \frac{1}{2}my^2\omega_z^2;$$

$$\frac{1}{2}mv_y'^2 = \frac{1}{2}m\bar{x}^2\omega_z^2 + m\bar{x}\,x\omega_z^2 + \frac{1}{2}mx^2\omega_z^2.$$

Adding these, we have for the sum of the kinetic energy of all the particles

$$E' = \Sigma\frac{1}{2}mv'^2$$

$$= \frac{1}{2}(\bar{x}^2 + \bar{y}^2)\omega_z^2\Sigma m - \bar{y}\omega_z^2\Sigma my + \bar{x}\omega_z^2\Sigma mx + \frac{1}{2}\omega_z^2\Sigma m(x^2 + y^2),$$

or, reducing by (3), (4) and (5),

$$E' = \frac{1}{2}M(\bar{x}^2 + y^2)\,\omega_z^2 + \frac{1}{2}I_z\omega_z^2 = \frac{1}{2}I_z'\omega_z^2. \qquad (11)$$

This is equation (IV), page 171.

Moment of the Effective Forces.—Let $\mathfrak{M}'_{fx}$, $\mathfrak{M}'_{fy}$, $\mathfrak{M}'_{fz}$ be the sums of the moments of the *effective forces* (page 168) about the co-ordinate axes OX', OY', OZ'. Then we have from (2), after reduction by (3), (4) and (5),

$$\left.\begin{aligned} \mathfrak{M}'_{fx} &= \Sigma m(f_z'y' - f_y'z') = -\alpha_z\Sigma mxz + \omega_z^2\Sigma myz;\\ \mathfrak{M}'_{fy} &= \Sigma m(f_x'z' - f_z'x') = -\alpha_z\Sigma myz - \omega_z^2\Sigma mxz;\\ \mathfrak{M}'_{fz} &= \Sigma m(f_y'x' - f_x'y') = I_z'\alpha_z. \end{aligned}\right\} \qquad (12)$$

The last of these equations is equation (I), page 170.

We have (page 191), reducing by (3) and (5),

$$\Sigma m(f'_{tz}y - f'_{ty}z) = -\alpha_z\Sigma mxz;$$

$$\Sigma m(f'_{tx}z - f'_{tz}x) = -\alpha_z\Sigma myz;$$

$$\Sigma m(f'_{ty}x - f'_{tx}y) = I_z\alpha_z.$$

These terms in equations (12) therefore give the moments about CX, CY, CZ of the effective tangential forces. We have also from (1*a*), reducing by (3),

$$\Sigma m(f'_{nz}y - f'_{ny}z) = +\omega_z^2\Sigma myz, \quad \Sigma m(f'_{nx}z - f'_{nz}x) = -\omega_z^2\Sigma mxz.$$

These terms in exquations (12) therefore give the moments about CX, CY of the effective deflecting forces.

If the fixed axis OZ' is a principal axis we have (page 192)

$$\mathfrak{M}'_{fx} = 0, \qquad \mathfrak{M}'_{fy} = 0, \qquad \mathfrak{M}'_{fz} = I_z'\alpha_z.$$

External Forces.—If any two points of the body situated upon the axis of rotation are fixed the axis is fixed. Conceive then the body to be fixed to the axis at two points distant a' and a'' from the origin O, and let the reactions of these points on the body resolved parallel to the co-ordinate axes be respectively R_x', R_y', R_z' and R_x'', R_y'', R_z'' (see figure, page 190).

These forces are *impressed forces* (page 168); but since they are internal to the system, consisting of the body and some other body upon which the axis rests or to which it is fastened, we call them *internal forces.*

All *other* impressed forces acting upon the body we may then call *external forces.*

Let these other impressed external forces be F_1, F_2, F_3, etc., making the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc., with the co-ordinate axes. Then we have for the resultant components of these external forces

$$\left.\begin{aligned} F_x &= F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + \text{etc.} = \Sigma F\cos\alpha; \\ F_y &= F_1 \cos\beta_1 + F_2 \cos\beta_2 + \text{etc.} = \Sigma F\cos\beta; \\ F_z &= F_1 \cos\gamma_1 + F_2 \cos\gamma_2 + \text{etc.} = \Sigma F\cos\gamma. \end{aligned}\right\} \quad . \quad . \quad (13)$$

Moment of the External Forces. — Let $\mathfrak{M}'_{ex}$, $\mathfrak{M}'_{ey}$, $\mathfrak{M}'_{ez}$ be the sums of the moments of the external forces about the co-ordinate axes OX', OY', OZ', and let (x_1', y_1', z_1'), (x_2', y_2', z_2'), etc., be the co-ordinates of the points of application of the external forces F_1, F_2, etc. Then we have

$$\left.\begin{aligned} \mathfrak{M}'_{ex} &= \Sigma Fy'\cos\gamma - \Sigma Fz'\cos\beta; \\ \mathfrak{M}'_{ey} &= \Sigma Fz'\cos\alpha - \Sigma Fx'\cos\gamma; \\ \mathfrak{M}'_{ez} &= \Sigma Fx'\cos\beta - \Sigma Fy'\cos\alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Pressures on the Fixed Axis.—We have by D'Alembert's principle (page 168) the resultant of the impressed forces equal to the resultant of the effective forces, or, from (6),

$$\left.\begin{aligned} F_x + R_x' + R_x'' &= \Sigma mf_x' = -M\bar{y}\alpha_z - M\bar{x}\omega_z^2 = M\overline{f_x}; \\ F_y + R_y' + R_y'' &= \Sigma mf_y' = M\bar{x}\alpha_z - M\bar{y}\omega_z^2 = M\overline{f_y}; \\ F_z + R_z' + R_z'' &= \Sigma mf_z' = 0. \end{aligned}\right\} \qquad (15)$$

Also taking moments about the co-ordinate axes, we have by D'Alembert's principle, from (12) and (14),

$$\left.\begin{aligned} &\Sigma Fy'\cos\gamma - \Sigma Fz'\cos\beta - R_y'a' - R_y''a'' \\ &\qquad\qquad = \mathfrak{M}'_{fx} = -\alpha_z\Sigma mxz + \omega_z^2\Sigma myz; \\ &\Sigma Fz'\cos\alpha - \Sigma Fx'\cos\gamma + R_x'a' + R_x''a'' \\ &\qquad\qquad = \mathfrak{M}'_{fy} = -\alpha_z\Sigma myz - \omega_z^2\Sigma mxz; \\ &\Sigma Fx'\cos\beta - \Sigma Fy'\cos\alpha \quad = \mathfrak{M}'_{fz} = I_z'\alpha_z. \end{aligned}\right\} \qquad (16)$$

From the last of these equations we can find α_z, and then from the first two and the first two of (15) we can find R_x', R_x'', R_y', R_y''. Then

R_z' and R_z'' are indeterminate, but their sum is given by the last of equations (15).

If there are no external forces, or if all the *impressed* forces pass through *the centre of mass*, we have from the last of equations (16) in either case $\alpha_z = 0$, and all terms containing α_z in (15) and (16) disappear. Now $-M\bar{x}\omega_z^2$ and $-M\bar{y}\omega_z^2$ are the sums of the components parallel to X' and Y' of the deflecting forces of the particles and $+\omega_z^2\Sigma my'z'$ and $-\omega_z^2\Sigma mx'z'$ are the moments about X' and Y' of the deflecting forces of the particles.

If OZ' and hence *CZ is a principal axis*, we have, taking the other two principal axes as the co-ordinate axes X' and Y' (page 192), $\Sigma myz = 0$, $\Sigma mxz = 0$, or the moments of the deflecting forces are zero.

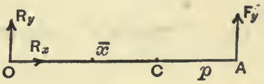
If the fixed axis passes through the centre of mass, we have in (15) $\bar{x} = 0$, $\bar{y} = 0$, or the sums of the components of the deflecting forces $M\bar{x}\omega_z^2$ and $M\bar{y}\omega_z^2$ are zero.

Hence, *if a body rotates about a principal fixed axis through the centre of mass, there will be no stress on that axis* **due to the deflecting forces.**

Permanent Axis.—If, then, *there are no external forces*, or if all the *impressed* forces pass *through the centre of mass* and a free body rotates about a principal axis through the centre of mass, that axis remains unchanged in direction in space and the body will always rotate about it with uniform angular velocity. For this reason it is called an axis of permanent rotation, or the **permanent axis.**

If there are no external forces, or if all the impressed forces pass through the centre of mass, and a free body rotates about some other axis than the principal axis through the centre of mass, the deflecting forces of the particles will cause the axis of rotation to change its direction and the body will never rotate about the permanent axis. If, therefore, we observe a body to rotate a short time about an unchanging axis with uniform angular velocity, we infer that it rotated about it from the beginning of the motion, that the axis is a principal axis through the centre of mass and that all the impressed forces are either zero or pass through the centre of mass.

Centre of Percussion.—Suppose a single force acting in the plane of rotation at right angles to OC. Take OC as the axis of X', and let F_y be the force, and the distance $CA = p$. Then $F_x = 0$, $F_z = 0$, $R_z' = 0$, $R_z'' = 0$, $\bar{y} = 0$, $z' = 0$, $x' = \bar{x} + p$, $y' = 0$, $\cos\alpha = 0$, $\cos\beta = 1$, $\cos\gamma = 0$. Hence, from (15) and (16),



$$R_x = R_x' + R_x'' = -M\bar{x}\omega_z^2;$$

$$R_y = R_y' + R_y'' = -F_y + M\bar{x}\alpha_z;$$

$$\mathfrak{M}'_{fx} = 0, \quad \mathfrak{M}'_{fy} = 0, \quad \mathfrak{M}'_{fz} = F_y(\bar{x} + p) = I_z'\alpha_z = M(\kappa^2 + \bar{x}^2)\alpha_z.$$

If ω_z is zero, R_x is zero. If we eliminate α_z from the second and last of these equations we obain

$$R_y = \frac{F_y(p\bar{x} - \kappa^2)}{\kappa^2 + \bar{x}^2}.$$

This is the same as equation (3), page 181. For $R_y = 0$, then, we have for the centre of percussion, just as on page 181,

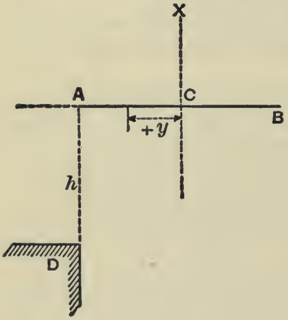
$$p = \frac{\kappa^2}{\bar{x}} = \frac{I_z}{M\bar{x}}, \quad \text{or} \quad p + \bar{x} = \frac{\kappa^2 + \bar{x}^2}{\bar{x}} = \frac{I_z'}{M\bar{x}}.$$

EXAMPLES.

(The student should carefully check these examples.)

(1) *A prismatic bar AB falls through a height h, retaining its horizontal position until one end strikes a fixed obstacle D. Find the motion after impact, considering the bodies non-elastic.*

Ans. Let m_2 be the mass of the bar, l its length, and u_2 the velocity of the centre of mass C at the instant of impact.



Then in equations (4), (5), (6), page 186, we have

$$e = 0,\ \epsilon_2 = 0,\ m_1 = \infty = m_1 + m_2,\ u_1 = 0.$$

Hence we have for the velocity of translation of AB after impact

$$v_2 = u_2 - u_2\frac{\kappa_z^2}{\kappa_z^2 + p^2} = \frac{u_2 p^2}{\kappa_z^2 + p^2}, \quad . \quad (1)$$

and for the angular velocity about C after impact

$$\omega_2 = \frac{u_2 p}{\kappa_z^2 + p^2}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the present case we have (page 177) $\kappa_z^2 = \frac{l^2}{12}$, and for the end A striking $p = +\frac{l}{2}$, for the end B striking $p = -\frac{l}{2}$, and $u_2 = -\sqrt{2gh}$, the minus sign denoting motion towards D. Hence in both cases of A or B striking

$$v_2 = -\frac{3}{4}\sqrt{2gh},$$

or the motion of C is towards D. Also if A strikes

$$\omega_2 = -\frac{3}{2l}\sqrt{2gh},$$

or if B strikes

$$\omega_2 = +\frac{3}{2l}\sqrt{2gh},$$

the (+) sign denoting counter-clockwise and the (−) sign clockwise rotation about C.

We can obtain (1) and (2) directly as follows: From page 171 we have

$$\omega_2 = \frac{\text{moment of momentum}}{I'} = \frac{m_2 u_2 p}{m_2(\kappa_z^2 + p^2)} = \frac{u_2 p}{\kappa_z^2 + p^2}.$$

Also at the instant of impact

$$v_2 = p\omega_2.$$

Hence

$$v_2 = \frac{u_2 p^2}{\kappa_z^2 + p^2}.$$

The momentum after impact is then

$$m_2v_2 = \frac{m_2u_2p^2}{\kappa_z^2 + p^2} = -\frac{3}{4}m_2\sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The impulse is

$$m_2(v_2 - u_2) = -\frac{m_2u_2\kappa_z^2}{\kappa_z^2 + p^2} = \frac{1}{4}m_2\sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The velocity at any point distant y from C, after impact, is

$$v_2 - y\omega_2 = \frac{u_2p}{\kappa_z^2 + p^2}(p - y) = -\frac{p\sqrt{2gh}}{\kappa_z^2 + p^2}(p - y), \quad . \quad . \quad . \quad (5)$$

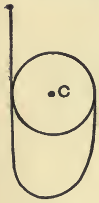
where p and y are positive towards A and negative towards B. Hence for A striking

$$v_2 - y\omega_2 = -\frac{3\sqrt{2gh}}{2l}\left(\frac{l}{2} - y\right),$$

and for B striking

$$v_2 - y\omega_2 = -\frac{3\sqrt{2gh}}{2l}\left(\frac{l}{2} + y\right).$$

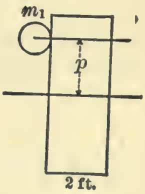
After impact the centre moves in the same vertical with a uniform acceleration g, while the angular velocity ω remains unchanged.



(2) *An inextensible string is wound around a cylinder and has its free end attached to a fixed point. The cylinder falls through a height h, and at the instant of impact the string is vertical and tangent to the cylinder. Find the motion after impact.*

Ans. $v = \dfrac{ur^2}{\kappa^2 + r^2} = \dfrac{2}{3}u, \quad \omega = \dfrac{ur}{\kappa^2 + r^2} = \dfrac{2}{3}\dfrac{u}{r}.$

(3) *An iron ball of mass $m_1 = 65$ pounds moving with a velocity of 36 ft. per sec. strikes a pine beam of uniform rectangular cross-section in the centre line of a side, at right angles, at a distance $p = 1\frac{3}{4}$ ft. above the centre of mass. The mass of the beam is $m_2 = 842.4$ pounds, its length 5 ft. and breadth 2 ft. If the beam is at rest, find the motion after impact, considering the impact as non-elastic.*



Ans. The moment of inertia (page 175) is the same as for $\frac{m_2}{3}$ concentrated at a corner. We have then

$$I = \frac{m_2}{3}\left[\left(\frac{5}{2}\right)^2 + \left(\frac{2}{2}\right)^2\right] = 2.416m_2, \quad \text{or} \quad \kappa_z^2 = 2.416.$$

Hence, from page 186, the velocity of the ball after impact is

$$v_1 = u_1 - \frac{m_2\kappa_z^2u_1}{(m_1 + m_2)\kappa_z^2 + m_1p^2} = 36\left(1 - \frac{2035.2}{2192.3 + 65\times1.75}\right) = 5.364 \text{ ft. per sec.}$$

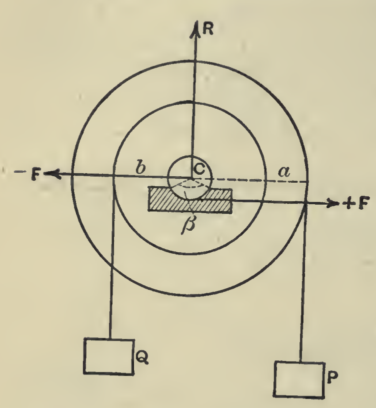
The velocity of the centre of mass of the beam after impact is

$$v_2 = \frac{m_1\kappa_z^2u_1}{(m_1 + m_2)\kappa_z^2 + m_1p^2} = 2.364 \text{ ft. per sec.}$$

The angular velocity of the beam after impact is

$$\omega_2 = -\frac{m_1 p u_1}{(m_1 + m_2)\kappa_z^2 + m_1 p^2} = -1.712 \text{ radians per sec.}$$

(4) *Suppose a wheel and axle composed of hollow disks for the wheel and axle and a solid cylinder for the journal. The radius of the wheel is $a = 3$ ft., of the axle $b = 2$ ft., of the journal $r = 1$ inch. Let the mass of the wheel be $W = 5$ lbs., of the axle $A = 3$ lbs., of the journal $J = 2$ lbs. Let the moving mass be $P = 10$ lbs. and the mass lifted $Q = 5$ lbs. Let the string be perfectly flexible and disregard its mass. Let P start from rest and fall for a time $t = 5$ sec. Discuss the motion of the apparatus, taking into account the mass of the wheel, axle and journal, and the friction, the coefficient of kinetic friction being $\mu = 0.07$. Take $g = 32\frac{1}{6}$ ft.-per-sec. per sec.* (*Compare example* (7), *page* 79.)



Ans. From page 176 we have for axis through the centre of mass C of wheel, axle and journal, at right angles to the plane of the wheel,

$$\text{Moment of inertia of wheel} = \frac{W}{2}(a^2 + b^2) = 32.5 \text{ lb.-ft.}^2;$$

$$\text{“ “ “ “ axle} = \frac{A}{2}(b^2 + r^2) = 6\tfrac{1}{96} \quad \text{“}$$

$$\text{“ “ “ “ journal} = \frac{J}{2}r^2 = \frac{1}{144} \quad \text{“}$$

Hence the moment of inertia of wheel, axle and journal is

$$I = \frac{W}{2}(a^2 + b^2) + \frac{A}{2}(b^2 + r^2) + \frac{J}{2}r^2 = 38\tfrac{149}{288} \text{ lb.-ft.}^2$$

Let f be the linear acceleration at the circumference of the wheel, and α its angular velocity. Then we have

$$a\alpha = f, \quad \text{or} \quad \alpha = \frac{f}{a},$$

and the linear acceleration at the circumference of the axle is

$$b\alpha = \frac{b}{a}f.$$

Now, by D'Alembert's principle, page 168, the impressed forces and the *reversed effective* forces constitute a system of forces in equilibrium. That is, the algebraic sum of the horizontal and vertical components must be zero, and the algebraic sum of the moments of the forces about any point must be zero.

The impressed forces are the upward reaction R at the centre C, the downward weights Pg, Qg, Wg, Ag, Jg of the masses P, Q, and the wheel, axle and journal, the friction F at the circumference of the journal, and the equal opposite and parallel reaction $-F$ of the bearing. The moment of the friction is then the moment of a couple $+F$, $-F$, or is Fr at any point (page 72, Vol. II, *Statics*).

The effective forces are Pf acting down, $Q\frac{b}{a}f$ acting up and the effective forces of the particles of the wheel, axle and journal. All these effective forces are to be reversed in direction. Since C is the centre of mass of the wheel, axle and journal, the effective forces of the particles occur in couples, and their algebraic sum is zero, and hence the algebraic sum of their components in any direction is zero. The algebraic sum of their moments about C is, from page 170, given by

$$I\alpha = I\frac{f}{a} = \left[\frac{W}{2a}(a^2+b^2) + \frac{A}{2a}(b^2+r^2) + \frac{J}{2a}r^2\right]f = 12\tfrac{725}{864}f \text{ poundal-ft.}$$

We have then for equilibrium of the impressed forces and the reversed effective forces, taking forces to the right and upwards positive and forces to the left and downwards negative,

$$+F - F = 0; \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$+R - Pg - Qg - Wg - Ag - Jg + Pf - Q\frac{b}{a}f = 0; \quad \ldots \quad (2)$$

and taking moments about C and counter-clockwise rotation positive,

$$-Pga + Qgb + Fr + I\alpha + Pfa + Q\frac{b^2}{a}f = 0. \quad \ldots \quad (3)$$

From (2) we have for the pressure upon the bearing

$$R = (P+Q+W+A+J)g - \left(P - Q\frac{b}{a}\right)f \text{ poundals.}$$

We can also find R as we have in Ex. (7), page 79. Thus,

$$\text{Tension on left} = Q\left(g + \frac{b}{a}f\right) \text{ poundals.}$$

$$\text{" " right} = P(g-f) \qquad \text{"}$$

Hence pressure on bearing is

$$R = (W+A+J)g + Q\left(g+\frac{b}{a}f\right) + P(g-f),$$

or

$$R = (P+Q+W+A+J)g - \left(P - Q\frac{b}{a}\right)f \text{ poundals.}$$

The friction for *new bearing* (page 79) is then

$$F = \frac{\mu\beta}{\sin\beta}R = \frac{\mu\beta}{\sin\beta}\left[(P+Q+W+A+J)g - \left(P - Q\frac{b}{a}\right)f\right] \text{poundals,}$$

where μ is the coefficient of kinetic friction and β is the angle of bearing.

Inserting this value of F and the value of $I\alpha$ in (3), we obtain for the acceleration f at the circumference of the wheel (compare Ex. (7), page 79)

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{\mu r\beta}{a\sin\beta}(P+Q+W+A+J)g}{\left[P + Q\frac{b^2}{a^2} + \frac{W}{2a^2}(a^2+b^2) + \frac{A}{2a^2}(b^2+r^2) + \frac{J}{2a^2}r^2\right] - \frac{\mu r\beta}{a\sin\beta}\left(P - Q\frac{b}{a}\right)}. \quad (4)$$

If β is small, $\sin \beta = \beta$, and we have for the given numerical values

$$f = 0.40136g = 12.912 \text{ ft.-per-sec. per sec.}$$

The acceleration of Q is then

$$\frac{b}{a}f = 8.608 \text{ ft.-per-sec. per sec.}$$

The velocity of P at the end of the time $t = 5$ sec. is

$$v = ft = 64.56 \text{ ft. per sec.},$$

and the angular velocity of the wheel is

$$\omega = \frac{v}{a} = 21.52 \text{ radians per sec.}$$

The velocity of Q at the end of $t = s$ sec. is

$$\frac{b}{a}v = 43.04 \text{ ft. per sec.}$$

The pressure R on the bearing is

$$R = 22.32426g \text{ poundals} = 22.32426 \text{ pounds.}$$

The friction is

$$F = \mu R = 1.5627g \text{ poundals} = 1.5627 \text{ pounds.}$$

The moment of the friction is

$$Fr = 0.13022g \text{ poundal-ft.} = 0.13022 \text{ pound-ft.}$$

The moment of the effective forces of the particles of the wheel, axle and journal is

$$I\alpha = 5.1531g \text{ poundal-ft.} = 5.1531 \text{ pound-ft.}$$

The distance s described by P is

$$s = \frac{1}{2}ft^2 = 161.4 \text{ ft.}$$

The distance described by Q is

$$\frac{b}{a}s = 107.6 \text{ ft.}$$

Tension on right $= P(g - f)$ $= 5.9864g$ poundals $= 5.9864$ pounds.
‘ “ left $= Q\left(g + \frac{b}{a}f\right) = 6.3378g$ “ $= 6.3378$ “

Moment of tension on right $= 17.9592$ pound-ft.
“ “ “ “ left $= \underline{12.6756}$ “
Difference $= 5.2836 = Fr + I\alpha$.

Work of $P = 5.9864 \times 161.4 = 966.205$ ft.-lbs. $= Ps - \dfrac{Pv^2}{2g}$.
“ on $Q = 6.3378 \times 107.6 = \underline{681.947}$ “ $= Q\dfrac{b}{a}s + \dfrac{b^2Qv^2}{2a^2g}$.

Work of friction and on wheel $= 284.258$ “

Work of friction and on wheel = 284.258 ft.-lbs.

Work of friction $= F\frac{r}{a}s$ $\quad = \quad$ 7.005 "

$$\text{Work on wheel} = 277.253 = \frac{1}{2}\frac{I}{g}\omega^2 \text{ (page 171).}$$

The *power* of P (page 49) $= \frac{966.205}{5} = 193.241$ ft.-lbs. per sec., or

$$\frac{193.241}{550} = 0.35 \text{ horse-power.}$$

The efficiency of the machine (page 52) is

$$\epsilon = \frac{681.947}{966.205} = 0.70.$$

From equation (3) we see that $I\alpha + Pfa + Q\frac{b^2}{a}f$ is the sum of the moments of the reversed effective forces. From page 174 this is equal to Mfa, where M is the *reduced mass* of P, Q, W, A and J, *reduced to the circumference of the wheel.* This reduced mass (page 174) is

$$M = P + Q\frac{b^2}{a^2} + \frac{W}{2a^2}(a^2 + b^2) + \frac{A}{2a^2}(b^2 + r^2) + \frac{J}{2a^2}r^2.$$

We can then write, instead of (3),

$$-Pga + Qgb + Fr + Mfa = 0,$$

or

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{r}{a}F}{M} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \tag{5}$$

If we substitute the values of M and F we obtain (4).

Now Pg is the weight of P, and $Q\frac{b}{a}g$ is the weight of Q *reduced to the circumference of the wheel*, that is, is the weight which acting at the circumference would have the same moment as Qg acting where it does. In the same way $\frac{r}{A}F$ is the friction reduced to the circumference of the wheel.

Hence $Pg - Q\frac{b}{a}g - \frac{r}{a}F$ is the *reduced moving force.* We have then the equation of force (page 2)

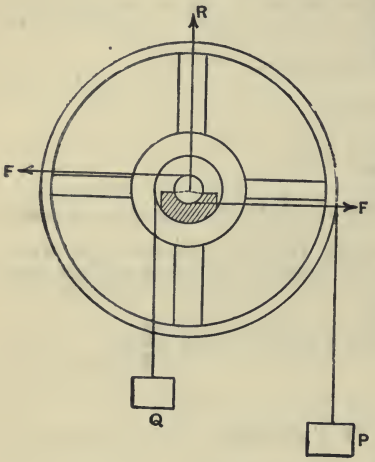
$$Mf = \textit{reduced moving force},$$

or

$$f = \frac{\text{reduced moving force}}{\text{reduced mass}} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \tag{6}$$

(5) *Suppose a wheel and axle composed of hollow disks for the rim or outer circumference C, the hub H and the axle A, of a solid cylinder for the journal J, and of four spokes, each spoke S being a bar of uniform cross-section. Let the outer radius of C be $a = 20$ inches and the inner radius $r_4 = 19$ inches, the outer radius of H be $r_1 = 8$ inches and the inner radius $b = 6$ inches, the radius of the journal J be $r = 1$ inch. Let the mass of the rim or outer circumference be $C = 40$ lbs., of the hub $H = 12$ lbs., of the axle $A = 10$ lbs., of the journal $J = 2$ lbs., and of each spoke $S = 15/4$ lbs. Let*

the moving mass be $P = 60$ *lbs. and the mass lifted* $Q = 160$ *lbs. Let the string be perfectly flexible and disregard its mass. Let* P *start from rest and fall for a time* $t = 3$ *sec. Discuss the motion of the apparatus, taking into account the mass of the wheel, axle, journal and spokes, and the friction, the coefficient of kinetic friction being* $\pi = 0.07$. *Take* $g = 32\frac{1}{6}$ *ft.-per-sec. per sec.*



Ans. The moment of inertia of a spoke with reference to an axis through its centre of mass at right angles to the plane of the wheel is (page 175)

$$\frac{S}{3}\left(\frac{r_4 - r_1}{2}\right)^2.$$

With reference to a parallel axis through the centre C it is then (page 173)

$$\frac{S}{3}\left(\frac{r_4 - r_1}{2}\right)^2 + S\left(\frac{r_4 + r_1}{2}\right)^2.$$

For four spokes we have then the moment of inertia for the axis through C at right angles to the plane of the wheel

$$\frac{4S}{3}\left(\frac{r_4 - r_1}{2}\right)^2 + 4S\left(\frac{r_4 + r_1}{2}\right)^2.$$

The reduced mass of the spokes, reduced to the circumference, is then (page 174)

$$\frac{4S}{3a^2}\left(\frac{r_4 - r_1}{2}\right)^2 + \frac{4S}{a^2}\left(\frac{r_4 + r_1}{2}\right)^2 = 7.2125 \text{ lbs.}$$

The moment of inertia of the rim is (page 176)

$$\frac{C}{2}(a^2 + r_4^2),$$

and the reduced mass of the rim is then

$$\frac{C}{2a^2}(a^2 + r_4^2) = 38.05 \text{ lbs.}$$

The moment of inertia of the hub is

$$\frac{H}{2}(r_1^2 + b^2),$$

and the reduced mass of the hub is

$$\frac{H}{2a^2}(r_1^2 + b^2) = 1.5 \text{ lbs.}$$

The moment of inertia of the axle is

$$\frac{A}{2}(b^2 + r^2),$$

and the reduced mass of the axle is

$$\frac{A}{2a^2}(b^2 + r^2) = 0.4625 \text{ lbs.}$$

The moment of inertia of the journal is (page 175)

$$\frac{Jr^2}{2},$$

and the reduced mass of the journal is

$$\frac{Jr^2}{2a^2} = 0.00125 \text{ lbs.}$$

The reduced mass of Q is

$$\frac{Qb^2}{a^2} = 14.4 \text{ lbs.}$$

The reduced mass of P is $P = 60$ pounds.

Hence the total reduced mass is

$$M = 60 + 14.4 + 0.00125 + 0.4625 + 1.5 + 38.05 + 7.2125 = 121.62625 \text{ lbs.}$$

From the preceding example we have for the acceleration f of P

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{r}{a}F}{M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the pressure R upon the journal we have, just as in the preceding example,

$$R = (P + Q + C + 4S + H + A + J)g - \left(P - Q\frac{b}{a}\right)f \text{ poundals.}$$

The friction for new bearing (page 79) is then

$$F = \frac{\mu\beta}{\sin\beta}R = \frac{\mu\beta}{\sin\beta}\left[\left(P+Q+C+4S+H+A+J\right)g - \left(P - Q\frac{b}{a}\right)f\right] \text{ pndls.,}$$

where μ is the coefficient of kinetic friction and β is the angle of bearing.

Inserting this value of F in (1), we obtain

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{\mu r\beta}{a\sin\beta}(P + Q + C + 4S + H + A + J)g}{M - \frac{\mu r\beta}{a\sin\beta}\left(P - Q\frac{b}{a}\right)}$$

If β is small, $\sin\beta = \beta$ and we have for the given numerical values

$$f = 0.09g = 2.895 \text{ ft.-per-sec. per sec.}$$

The acceleration of Q is then

$$\frac{b}{a}f = 0.8685 \text{ ft.-per-sec. per sec.}$$

The velocity of P at the end of the time $t = 3$ sec. is

$$v = ft = 8.685 \text{ ft. per sec.,}$$

and the angular velocity of the wheel is

$$\omega = \frac{v}{a} = 5.211 \text{ radians per sec.}$$

The velocity of Q at the end of $t = 3$ sec. is

$$\frac{b}{a}v = 2.6055 \text{ ft. per sec.}$$

The pressure R on the bearing is

$$R = 297.92g \text{ poundals} = 297.92 \text{ lbs.}$$

The friction is

$$F = \mu R = 20.8544g \text{ poundals} = 20.8544 \text{ lbs.}$$

The moment of the friction is

$$Fr = 1.737875g \text{ poundal-ft.} = 1.737875 \text{ lb.-ft.}$$

The angular acceleration is

$$\alpha = \frac{f}{a} = 1.737 \text{ radians-per-sec. per sec.}$$

The moment of inertia for rim, spokes, hub, axle and journal is

$$I = 131.184 \text{ lb.-ft.}^2.$$

The moment of the effective forces of the particles of rim, spokes, hub, axle and journal is

$$I\alpha = 227.8666 \text{ poundal-ft.} = 7.0829 \text{ lb.-ft.}$$

The distance s described by P is

$$s = \frac{1}{2}ft^2 = 13.0275 \text{ ft.}$$

The distance described by Q is

$$\frac{b}{a}s = 3.90825 \text{ ft.}$$

$$\begin{array}{llll}
\text{Tension on right} = P(g-f) & = 54.6g \text{ poundals} & = & 54.6 \text{ lbs.} \\
\text{“ \quad “ \quad left} = Q\left(g + \frac{b}{a}f\right) & = 164.32g \quad \text{“} & = & 164.32 \text{ “}
\end{array}$$

$$\begin{array}{rl}
\text{Moment of tension on right} = & 91.00 \text{ lb.-ft.} \\
\text{“ \quad “ \quad “ \quad “ left} = & 82.16 \text{ “ \ “} \\
\hline
\text{Difference} = & 8.84 = Fr + I\alpha.
\end{array}$$

$$\begin{array}{lll}
\text{Work of } P = 54.6 \times 13.0275 & = 711.3015 \text{ ft.-lbs.} & = Ps - \frac{Pv^2}{2g}. \\
\text{“ \quad on } Q = 164.32 \times 3.90825 & = 642.2036 \quad \text{“} & = Q\frac{b}{a}s + \frac{b^2Qv^2}{2a^2g}. \\
\hline
\text{“ \quad of friction and on wheel} & = 69.0979 \quad \text{“} & \\
\text{“ \quad of friction} = F\frac{r}{a}s & = 13.5841 \quad \text{“} & \\
\hline
\text{Work on wheel} & = 55.5138 \quad \text{“} & = \frac{1}{2}\frac{I}{g}\omega^2 \text{ (page 171).}
\end{array}$$

The power of P (page 49) $= \frac{711.3015}{3} = 237.1005$ ft.-lbs. per sec., or

$$\frac{237.1005}{550} = 0.43 \text{ horse-power.}$$

The efficiency of the machine (page 52) is

$$\epsilon = \frac{642.2036}{711.3015} = 0.93.$$

(6) *A hollow circular disk whose outer radius is a_1, inner radius b_1 and thickness t_1 revolves about an axis perpendicular to its plane. Find the thickness t_2 of an equivalent disk whose outer radius is a_2 and inner radius b_2.*

Ans. For any angular velocity ω or acceleration α we must have

$$I_1\omega \text{ or } I_1\alpha = I_2\omega \text{ or } I_2\alpha.$$

That is, $I_1 = I_2$. But (page 176)

$$I_1 = M_1(a_1^2 + b_1^2) = \delta\pi t_1(a_1^2 - b_1^2)(a_1^2 + b_1^2) = \delta\pi t_1(a_1^4 - b_1^4),$$

where δ is the density or mass of a unit of volume. In the same way

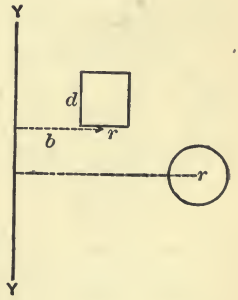
$$I_2 = \delta\pi t_2(a_2^4 - b_2^4).$$

Hence

$$t_2 = \frac{a_1^4 - b_1^4}{a_2^4 - b_2^4} \cdot t_1.$$

(7) *A sphere of radius r rotates about the axis YY at a distance a. Find the height d of an equivalent cylinder of radius of base r whose axis is parallel to YY at a distance b.*

Ans. The moment of inertia of a sphere whose mass is M_1 about any diameter is (page 176) $I_1 = \frac{2}{5}M_1r^2$. The moment of inertia of a cylinder of mass M_2 about its axis is (page 175) $M_2\frac{r^2}{2}$. With reference to the axis YY we have then



$$I_1' = \frac{2}{5}M_1r^2 + M_1a^2, \quad I_2' = M_2\frac{r^2}{2} + M_2b^2.$$

Hence we have

$$\frac{2}{5}M_1r^2 + M_1a^2 = M_2\frac{r^2}{2} + M_2b^2.$$

But if δ is the density,

$$M_1 = \frac{4\delta\pi r^3}{3} \quad \text{and} \quad M_2 = \delta\pi r^2 d.$$

Hence

$$d = \frac{8}{15}r \cdot \frac{2r^2 + 5a^2}{r^2 + 2b^2} \quad \ldots\ldots\ldots\ldots \quad (1)$$

If the cylinder and sphere rotate about the axis YY without turning on their own axes, they can be treated as particles, and we have

$$I_1' = M_1a^2, \quad I_2' = M_2b^2.$$

If the cylinder and sphere have the angular velocity or acceleration ω or α about their own axes and the angular velocity or acceleration ω' or α' about YY, either in the same or opposite directions, then we have

$$\frac{2}{5}M_1r^2\omega \pm M_1a^2\omega' = M_2\frac{r^2}{2}\omega \pm M_2b^2\omega',$$

or

$$\frac{2}{5}M_1r^2\alpha \pm M_1a^2\alpha' = M_2\frac{r^2}{2}\alpha \pm M_2b^2\alpha'.$$

Hence

$$d = \frac{8}{15}r\cdot\frac{2r^2\omega \pm 5a^2\omega'}{r^2\omega \pm 2b^2\omega'}, \quad \text{or} \quad d = \frac{8}{15}r\cdot\frac{2r^2\alpha \pm 5a^2\alpha'}{r^2\alpha \pm 2b^2\alpha'}. \quad . \quad . \quad . \quad (2)$$

If the bodies are rigidly connected with the axis YY we have $\omega = \omega'$ or $\alpha = \alpha'$ in the same direction, and obtain equation (1). If the bodies do not turn on their own axes ω and α are zero, and we have

$$d = \frac{8}{15}r\cdot\frac{5a^2}{2b^2} = \frac{4ra^2}{3b^2}.$$

If the bodies turn about their axes with the same angular velocity or acceleration, as about YY, but in the opposite direction, we have

$$d = \frac{8}{15}r\cdot\frac{2r^2 - 5a^2}{r^2 - 2b^2}.$$

(8) *Upon a vertical hollow axle whose outer radius is r_1 and inner radius r_2, and length l, there is fixed a circular disk of radius a, at right angles to the axle. Under the action of a force the angular velocity ω_1 is attained. If now the force ceases to act, find (a) the time of coming to rest; (b) the number of revolutions in that time.*

Ans. Let the mass of the axle be A, and of the disk D. Then the moment of inertia of the axle is (page 176)

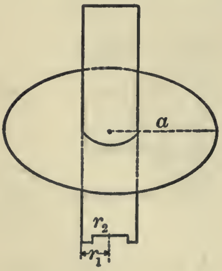
$$\frac{A}{2}(r_1^2 + r_2^2),$$

and the moment of inertia of the disk is

$$\frac{D}{2}(a^2 + r_1^2).$$

The total moment of inertia is then

$$I = \frac{A}{2}(r_1^2 + r_2^2) + \frac{D}{2}(a^2 + r_1^2).$$



The pressure on the axle is $(D + A)g$ poundals. The moment of the friction for hollow flat pivot is then (Vol. II, *Statics*, page 193)

$$M = \frac{2}{3}\mu(D + A)g\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right),$$

where μ is the coefficient of kinetic friction.

If α is the angular retardation, we have then

$$I\alpha = M, \quad \text{or} \quad \alpha = \frac{M}{I}.$$

The angular velocity at the end of any time t is then

$$\omega = \omega_1 - \alpha t.$$

The time of coming to rest is then

$$t = \frac{\omega_1}{\alpha} = \frac{I\omega_1}{M}.$$

The number of radians described in the time t is

$$\theta = \omega_1 t - \frac{1}{2}\alpha t^2 = \frac{I\omega_1^2}{2M}.$$

The number of revolutions is then

$$n = \frac{\theta}{2\pi} = \frac{I\omega_1^2}{4\pi M}.$$

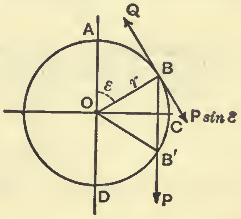
(9) *Find the mass of a fly-wheel for a given angular velocity, length of crank and applied force.*

Ans. Let the greatest angular velocity be ω_2, the least ω_1, and let ω be the mean angular velocity, and let the difference of ω_2 and α_1 be $\frac{\omega}{a}$. Then we have

$$\omega = \frac{\omega_2 + \omega_1}{2} \quad \text{and} \quad \frac{\omega}{a} = \omega_2 - \omega_1,$$

and hence

$$\omega_1 = \omega - \frac{\omega}{2a}, \quad \omega_2 = \omega + \frac{\omega}{2a}. \quad . \quad . \quad . \quad (1)$$



Let the length of crank be $r = OB$, and the connecting-rod be very long compared to r, so that the constant force P exerted by the connecting-rod may be considered as practically acting always in the same vertical direction. Let Q be the resistance at the end B of the crank.

When the crank has turned through the angle $AOB = \epsilon$ from the dead point A, the work of P is $Pr(1 - \cos\epsilon)$, and the work of the resistance is $Qr\epsilon$. If then the angular velocity at A is ω_1 and at B, ω_2, and I is the moment of inertia of the fly-wheel, we have for P and Q in pounds

$$\frac{1}{2g}I(\omega_2^2 - \omega_1^2) = Pr(1 - \cos\epsilon) - Qr\epsilon. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In every complete revolution the work of P and Q must be equal. We have then for a complete revolution,

(*a*) for single-acting engine

$$2rP = 2\pi rQ, \quad \text{or} \quad Q = \frac{1}{\pi}P = 0.3183P; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(*b*) for double-acting engine

$$4rP = 2\pi rQ, \quad \text{or} \quad Q = \frac{2}{\pi}P = 0.6366P. \quad . \quad . \quad . \quad . \quad (4)$$

(*a*) **Single-acting Engine.**—At any point B we can resolve P into a normal component along OB and a tangential component $P\sin\epsilon$, which causes change of motion. At the dead point A this component is zero and increases up to a point B for which it is equal to Q. We find the corresponding value of the angle $AOB = \epsilon_1$ then from

$$P\sin\epsilon_1 = Q = 0.3183P, \quad \text{or} \quad \epsilon_1 = 0.103\,\pi = 18^\circ\ 33'\ 36.5''.$$

From B, $P \sin \epsilon$ increases up to the point C, and from this point again decreases to the point B', where it is again equal to Q, and we have the corresponding value of the angle $AOB' = \epsilon_2$ from

$$P \sin \epsilon_2 = Q = 0.3183P, \quad \text{or} \quad \epsilon_2 = 0.897\,\pi = 161°\ 26'\ 23.5''.$$

From B', $P \sin \epsilon$ decreases to the dead point D, where it is again zero. The motion is then accelerated from B to B', since between these points $P \sin \epsilon$ is greater than Q. Between A and B and B' and D it is retarded. The angular velocity is then least at B and greatest at B', and then decreases to its minimum value at B again.

If the crank moves a fly-wheel the moment of inertia of which is I, we have then the increase of kinetic energy from B to B' equal to the work done, or for the distance BCB' and P and Q in pounds and r in feet

$$\frac{1}{2g}I(\omega_2^2 - \omega_1^2) = 2rP \cos \epsilon_1 - Qr(\pi - 2\epsilon_1). \quad . \quad . \quad . \quad . \quad (5)$$

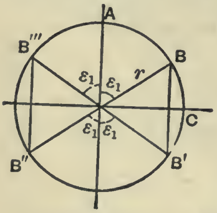
If we substitute in (5), $Q = \frac{1}{\pi}P$ from (3), we have

$$\frac{1}{2g}I(\omega_2^2 - \omega_1^2) = Pr\left(2 \cos \epsilon_1 - 1 + \frac{2\epsilon_1}{\pi}\right). \quad . \quad . \quad . \quad . \quad (6)$$

Substituting $\epsilon_1 = 0.103\,\pi$, $\cos \epsilon_1 = \sqrt{1 - \sin^2 \epsilon_1} = 0.948$ and the values of ω_1 and ω_2 from (1), we obtain

$$\left.\begin{aligned} I &= \frac{1.102 Prag}{\omega^2}, \\ \text{or} \\ I &= \frac{2.204 Prg}{\omega_2^2 - \omega_1^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (7) for a given force P in pounds, length of crank r in feet and range of angular velocity ω_1 and ω_2, or ratio $a = \frac{\omega_2 + \omega_1}{2(\omega_2 - \omega_1)}$, we can find the moment of inertia I of the fly-wheel and can then design it.



(*b*) **Double-acting Engine.**—At the point B we have, as before, from (4)

$$P \sin \epsilon_1 = Q = 0.6366P.$$

Hence

$$\epsilon_1 = 0.2196\,\pi = 39°\ 32'\ 19.5''.$$

From B to B' then the motion is accelerated as before, and the angular velocity is ω_1 at B and ω_2 at B'. From B' to B'' the motion is retarded to ω_1, from B'' to B''' accelerated to ω_2, and from B''' to B retarded to ω_1. We have then as before, for P and Q in pounds and r in feet,

$$\frac{1}{2g}I(\omega_2^2 - \omega_1^2) = 2Pr \cos \epsilon_1 - Qr(\pi - 2\epsilon_1).$$

Substituting as before, we obtain

$$\left.\begin{aligned} I &= \frac{0.4210 Prag}{\omega^2}; \\ I &= \frac{0.842 Prg}{\omega_2^2 - \omega_1^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We can call

$$a = \frac{\omega_2 + \omega_1}{2(\omega_2 - \omega_1)} = \frac{\omega}{\omega_2 - \omega_1}$$

the *coefficient of steadiness.* The greater a is taken, the less the difference $\omega_2 - \omega_1$ of the limiting velocities, and the steadier the action. Ordinarily a is taken at from 30 to 100, according to the steadiness desired.

If H is the horse-power of the engine and n the number of revolutions per minute, we have for single-acting engine

$$\frac{2Prn}{33000} = H, \quad \text{or} \quad Pr = \frac{16500H}{n}.$$

For double-acting engine

$$\frac{4Prn}{33000} = H, \quad \text{or} \quad Pr = \frac{8250H}{n}.$$

Thus for a double-acting engine of 25 horse-power making 32 revolutions per minute, and $a = 64$, we have $\omega = \frac{2\pi \times 32}{60}$ radians per sec. From (8), taking $g = 32$ ft.-per-sec. per sec.

$$I = 494900 \text{ lb.-ft.}^2$$

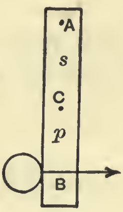
If the outside radius of the fly-wheel is $r_1 = 6$ ft. and the inside radius is $r_2 = 5.5$ ft., we have, if we disregard the spokes,

$$I = M(r_1^2 + r_2^2) = 494900, \quad \text{or} \quad M = 7470 \text{ lbs.}$$

If we take the density of iron, 480 lbs. per cubic foot, the thickness of the rim t, we have

$$M = 480 \times 2\pi r_1 t(r_1 - r_2) = 7470, \quad \text{or} \quad t = 0.8 \text{ ft.}$$

(10) *A homogeneous prismatic bar AB constrained to rotate about a fixed axis at A receives a direct impact from a sphere whose mass is m_1 and velocity u_1. Find the angular velocity ω_2 of the bar and the velocity v_1 of the sphere after impact if the bodies are perfectly elastic.*



Ans. Let the mass of the bar be m_2 and $\kappa_2'^2$ the square of its radius of gyration with reference to the axis at A. Also let a_2 be the distance AB.

Then from equations (4) and (3), page 183,

$$\omega_2 = -\frac{2m_1a_2u_1}{m_1a_2^2 + m_2\kappa_2'^2}, \quad v_1 = \frac{m_1a_2^2 - m_2\kappa_2'^2}{m_1a_2^2 + m_2\kappa_2'^2}u_1.$$

(11) *In the preceding example let there be no fixed axis. Find where the impact must take place in order that the initial velocity at the end A may be zero.*

Ans. At the centre of percussion (page 180). Hence if C is the centre of mass, κ_2' the radius of gyration with reference to the axis through A, $CA = s$ and $CB = p$, we have $p = \frac{\kappa^2}{s}$.

We obtain the same result from equations (5), (6), page 186.

Thus from (5), since $u_2 = 0$, $\epsilon_2 = 0$,

$$v_2 = \frac{(1 + e)m_1\kappa_2'^2u_1}{(m_1 + m_2)\kappa_2'^2 + m_1p^2},$$

and from (6)

$$\omega_2 = -\frac{(1+e)m_1 p u_1}{(m_1+m_2)\kappa_2'^2 + m_1 p^2}.$$

Now since for origin at A and AC coinciding with axis of Y, $v_2 = -s\omega_2$, we have

$$sp = \kappa_2'^2, \quad \text{or} \quad p = \frac{\kappa_2'^2}{s}.$$

We see then that the position of the point of impact is independent of the magnitude of the impulse and whatever the value of e, that is, whether the bodies are elastic or inelastic.

If the point of no initial velocity is at a distance d from C, we have $v_2 - d\omega = 0$, or $dp = -\kappa^2$, or $p = -\frac{\kappa^2}{d}$.

(12) *A horizontal uniform disk is free to revolve about a vertical axis through its centre. A man walks around on the outer edge. Find the angular distance passed over by the man and disk when he has walked once round the circumference.*

Ans. Let M be the mass of the man and D the mass of the disk, and r its radius. Then $I = \frac{D}{2}r^2$.

Let α be the angular acceleration of the disk and F the force exerted on the circumference. Then (page 170)

$$\alpha = \frac{Fr}{I} = \frac{2Fr}{Dr^2} = \frac{2F}{Dr}.$$

If α_1 is the angular acceleration of the mass, we have $F = Mr\alpha_1$, and hence

$$\alpha = \frac{2M}{D}\alpha_1.$$

The angular distance of the disk is $\frac{1}{2}\alpha t^2$ and of the mass $\frac{1}{2}\alpha_1 t^2$, and when the mass arises at the initial point we have

$$\frac{1}{2}\alpha t^2 + \frac{1}{2}\alpha_1 t^2 = 2\pi.$$

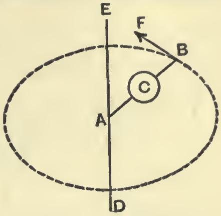
Inserting the value of α_1, we have for the angular distance of the mass

$$\frac{1}{2}\alpha_1 t^2 = \frac{2\pi D}{D+2M},$$

and for the angular distance of the disk

$$\frac{1}{2}\alpha t^2 = \frac{4\pi M}{D+2M}.$$

(13) *Let a body of mass M on the horizontal arm AB be free to rotate about the vertical axis ED. Let the body be acted upon by a horizontal force F of constant magnitude always at right angles to AB at the distance AB = r. Let the distance AC of the centre of mass C from the axis be d. Find the number of turns which the body will make about the axis DE in the time t.*



Ans. Let κ be the principal radius of gyration of the body with reference to the axis through C parallel to DE, and κ' the radius of gyration with reference to the axis DE. Then (page 176)

$$\kappa'^2 = \kappa^2 + d^2.$$

Then (page 170) we have for the angular acceleration

$$\alpha = \frac{Fr}{M(\kappa^2 + d^2)}.$$

If θ is the angular distance, we have $\theta = \frac{1}{2}\alpha t^2$, or

$$\theta = \frac{Frt^2}{2M(\kappa^2 + d^2)}.$$

The number of complete rotations will then be

$$n = \frac{\theta}{2\pi} = \frac{Frt^2}{4\pi M(\kappa^2 + d^2)}.$$

If the body is a sphere 2 feet in diameter, weighing 100 lbs., the centre 5 ft. from the axis, and F is a force of 25 lbs. at the end of a lever 8 feet long, find the number of turns in 5 minutes. ($g = 32$ *ft.-per-sec. per sec.*)

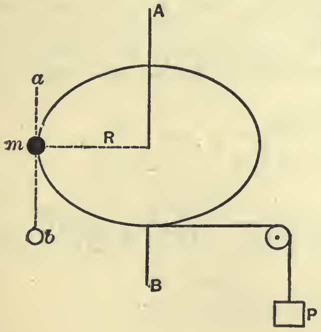
Ans. $$n = \frac{25g \times 8 \times 300^2}{4\pi \times 100\left(\frac{2}{5} + 25\right)} = \frac{7200000}{127\pi} = 1845\frac{45}{399} \text{ turns.}$$

The time necessary to make one turn is

$$t = \sqrt{\frac{4\pi \times 100\left(\frac{2}{5} + 25\right)}{25g \times 8}} = 2.23 \text{ sec.}$$

(14) *A sphere whose mass is m rests upon the rim of a horizontal disk of mass D. A perfectly flexible string passes round the disk and over a pulley and has a mass P attached to its lower end. Disregarding friction and the mass of the pulley and string, find the distance described by P in the time t.*

Ans. Let R be the radius of the disk and r the radius of the sphere.



If the sphere *moves with the disk as if it were part of it,* i.e., rotates about the axis ab in the same time that it rotates about the parallel axis AB, we have the moment of inertia of the sphere with reference to the axis AB, pages 173 and 176,

$$\frac{2}{5}mr^2 + mR^2.$$

In this case we have for the angular acceleration about AB

$$= \frac{PgR}{\frac{2}{5}mr^2 + mR^2 + \frac{D}{2}R^2}.$$

Hence the acceleration f of P is

$$f = R\alpha = \frac{Pg}{m\left(1 + \frac{2}{5}\frac{r^2}{R^2}\right) + \frac{D}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The distance described by P is then

$$s = \frac{1}{2}ft^2 = \frac{Pgt^2}{2m\left(1 + \frac{2r^2}{5R^2}\right) + D}.$$

If the sphere *does not rotate about the axis ab*, as when, for instance, it is hung from the rim, we may consider it as a particle, and its moment of inertia is mR^2. We have then

$$\alpha = \frac{PgR}{mR^2 + \frac{D}{2}R^2},$$

or

$$f = \frac{Pg}{m + \frac{D}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$s = \frac{Pgt^2}{2m + D}.$$

In either case, if we take the reduced mass (page 174), we have

Reduced mass × f = moving force.

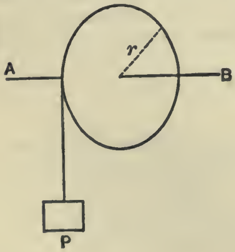
If the sphere has an angular acceleration α_1 not equal to α about ab in the positive direction (counter-clockwise), we have for the moment of the force causing this rotation, $\frac{2}{5}mr^2\alpha_1$. Hence, by D'Alembert's principle (page 168),

$$PgR - \frac{2}{5}mr^2\alpha_1 - mR^2\alpha - \frac{D}{2}R^2\alpha = 0,$$

or

$$\alpha = \frac{PgR - \frac{2}{5}mr^2\alpha_1}{mR^2 + \frac{D}{2}R^2}.$$

(15) *A disk of mass D is free to rotate about a horizontal axis AB. A perfectly flexible string passes round the disk and has a mass P attached to its lower end. Find the distance described by P in t seconds, neglecting friction and the mass of the string.*



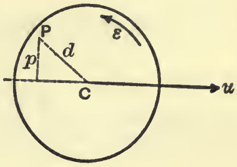
Ans. $\alpha = \frac{Pgr}{Pr^2 + \frac{D}{2}r^2}, \quad f = \frac{Pg}{P + \frac{D}{2}},$

$$s = \frac{1}{2}ft^2 = \frac{Pgt^2}{2P + D}, \qquad \theta = \frac{s}{r} = \frac{Pgt^2}{2Pr + Dr},$$

$$n = \frac{\theta}{2\pi} = \frac{Pgt^2}{2\pi(2P + D)r}.$$

(16) *A disk of mass D has a motion of translation u and of rotation ϵ in its own plane, when suddenly any point of the disk becomes fixed. Find the angular velocity ω about the fixed point.*

Ans. Let p be the perpendicular distance between the fixed point P and the direction of motion of translation u at the instant when P becomes fixed, and d the distance between P and C.



Then the moment of inertia of the disk with reference to the axis through P, if κ is the principal radius of gyration, is

$$D(\kappa^2 + d^2),$$

and the moment of momentum is

$$D\kappa^2\epsilon + Dup.$$

We have then (page 171)

$$\omega = \frac{D\kappa^2\epsilon + Dup}{D(\kappa^2 + d^2)} = \frac{\kappa^2\epsilon + up}{\kappa^2 + d^2}.$$

(17) *A sphere of mass m and radius r has an angular velocity ϵ and contracts until its radius is nr. Find the final angular velocity ω.*

Ans. By the principle of conservation of areas (page 142) the moment of momentum is constant Hence

$$\frac{2}{5}mr^2\epsilon = \frac{2}{5}mn^2r^2\omega, \quad \text{or} \quad \omega = \frac{1}{n^2}\epsilon^2.$$

The initial kinetic energy of rotation is (page 171)

$$E_1 = \frac{1}{5}mr^2\epsilon^2,$$

and the final kinetic energy of rotation is

$$E = \frac{1}{5}mn^2r^2\omega^2 = \frac{1}{5}m\frac{r^2}{n^2}\epsilon^2.$$

The gain of kinetic energy of rotation is then

$$E - E_1 = \frac{1}{5}mr^2\epsilon^2\left(\frac{1}{n^2} - 1\right).$$

This gain of kinetic energy must be at the expense of potential energy (page 87).

[(18) *In the preceding example find the loss of potential energy due to contraction.*

Ans. Let m' be the mass of a particle on the surface of the sphere. The attraction between the sphere and this particle is (Vol. II, *Statics*, page 47)

$$\kappa\frac{m'm}{r^2} = \frac{m'mR^2g}{Mr^2},$$

where R is the radius and M the mass of the earth, and g the acceleration of gravity at the earth's surface.

The attraction for any point within the sphere varies directly as the distance from the centre. Hence at a distance ρ from the centre the attraction is

$$\frac{m'mR^2g}{Mr^2} \cdot \frac{\rho}{r}.$$

During contraction the attraction is inversely as the square of the distance from the centre. Hence the attraction at a distance x of a particle originally at ρ is

$$\frac{m'mR^2g\rho}{Mr^3}\cdot\frac{\rho^2}{x^2}.$$

The loss of potential energy of the particle is then

$$\frac{m'mR^2g\rho^3}{Mr^3}\int_{x=n\rho}^{x=\rho}\frac{dx}{x^2}=\frac{m'mR^2g\rho^2}{Mr^3}\left(\frac{1-n}{n}\right).$$

The mass of a unit of volume of the sphere is $\dfrac{m}{\frac{4}{3}\pi r^3}$. The volume of a spherical shell of radius ρ is $4\pi\rho^2d\rho$. Hence the mass of an elementary shell is

$$m'=\frac{m}{\frac{4}{3}\pi r^3}\cdot 4\pi\rho^2d\rho=\frac{3m\rho^2d\rho}{r^3}.$$

Substituting this, we have for the loss of potential energy of an elementary shell

$$\frac{3m^2R^2g}{Mr^6}\left(\frac{1-n}{n}\right)\rho^4d\rho.$$

The total loss of potential energy is then

$$\frac{3m^2R^2g}{Mr^6}\left(\frac{1-n}{n}\right)\int_{\rho=0}^{\rho=r}\rho^4d\rho=\frac{3m^2R^2g}{5Mr}\left(\frac{1-n}{n}\right). \quad . \quad . \quad . \quad . \quad (1)$$

This loss of potential energy must be converted into kinetic energy (page 87).

We have just seen in the preceding example that the gain of kinetic energy of rotation is

$$\frac{1}{5}mr^2\epsilon^2\left(\frac{1-n^2}{n^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence if (1) is greater than (2), the difference must be converted into heat energy. The energy converted into heat is then

$$\frac{m(1-n)}{5Mn^2r}[3mR^2gn-Mr^3\epsilon^2(1+n)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we divide by g, we have this energy in ft.-lbs. If we then divide by J, the mechanical equivalent of heat, we obtain the number of heat units. We have then for the number of heat units generated

$$\text{No. of heat units}=\frac{m(1-n)}{5JMn^2r}\left[3mR^2n-\frac{Mr^3\epsilon^2(1+n)}{g}\right].$$

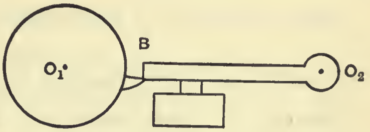
If δ is the density of the sphere and γ is the density of water, the mass of a volume of water equal to the sphere is $\frac{\delta}{\gamma}m$. If σ is the specific heat of the sphere and T the number of degrees rise of temperature, we have

$$\text{No. of heat units} = \frac{\sigma\delta}{\gamma}mT.$$

Hence

$$T = \frac{\gamma(1-n)}{5\sigma J\delta Mn^2r}\left[3mR^2n - \frac{Mr^3\epsilon^2(1+n)}{g}\right].$$

(19) *The moment of inertia of the shaft* O_1B *with reference to its axis of rotation is* $m_1\kappa_1^2 = 40000$ *lb.-ft.*^{2} *and that of the trip-hammer* BO_2 *with reference to its axis of rotation is* $m_2\kappa_2^2 = 150000$ *lb.-ft.*^{2} *The arm* O_1B *of the shaft is* $a_1 = 2$ *ft. and that* BO_2 *of the hammer is* $a_2 = 6$ *ft. The angular velocity of the shaft before impact is* $\epsilon_1 = 1.05$ *radians per sec. Find the velocity after impact and the loss of energy at each impact, supposing both bodies inelastic.*



Ans. (page 182). The angular velocity of the shaft after impact is

$$\omega_1 = 1.05 - \frac{4 \times 1.05 \times 150000}{40000 \times 36 + 150000 \times 4} = 0.741 \text{ radians per sec.}$$

The angular velocity of the hammer after impact is

$$\omega_2 = \frac{6 \times 2 \times 1.05}{51} = 0.247 \text{ radians per sec.}$$

The loss of energy at each impact is (page 171) in foot-poundals

$$\frac{m_1\kappa_1^2}{2}\epsilon_1^2 - \frac{m_1\kappa_1^2}{2}\omega_1^2 - \frac{m_2\kappa_2^2}{2}\omega_2^2.$$

In foot-pounds we have then

$$\frac{m_1\kappa_1^2}{2g}\epsilon_1^2 - \frac{m_1\kappa_1^2}{2g}\omega_1^2 - \frac{m_2\kappa_2^2}{2g}\omega_2^2 = 201.63 \text{ foot-pounds.}$$

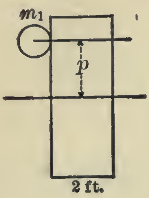
(20) *A ballistic pendulum weighing* 30000 *lbs. is set in oscillation by a 6-lb. ball, and the angular displacement is* 15°. *If the distance* s *of the centre of mass from the axis is* 5 *ft. and the distance* a_2 *of the point of impact below the axis is* 5.5 *ft., and the number of oscillations per minute is* $n = 40$, *find the velocity of the ball.*

Ans. (page 185). $u_1 = \frac{3006}{6} \cdot \frac{120 \times 32.2 \times 5}{40 \times 3.1416 \times 5.5} \sin 7\frac{1}{2}^\circ = 1828$ ft. per sec.

(21) *An iron ball of mass* $m_1 = 65$ *lbs. strikes with a velocity of* $u_1 = 36$ *ft. per sec. a beam of wood of rectangular cross-section whose mass is* $m_2 = 842.4$ *lbs. at a distance* $p = 1\frac{3}{4}$ *ft. above the centre of mass C. The length of the beam is* 5 *ft. and the thickness* 2 *ft. Find the velocity of the ball after impact, also the velocity of*

the centre of mass C and the angular velocity of the beam, regarding the bodies as inelastic.

Ans. (page 185). The square of the semi-diagonal is



$$\left(\frac{5}{2}\right)^2 + \left(\frac{2}{2}\right)^2 = 7.25.$$

Hence the moment of inertia of the beam (page 175) is

$$m^2\kappa^2 = \frac{m_2}{3} \times 7.25 = 2035.2$$

and

$$\kappa^2 = \frac{1}{3} \times 7.25 = 2.416.$$

Hence the velocity of the ball after impact is

$$v_1 = 36 - \frac{36 \times 2035.2}{907.4 \times 2.416 + 65 \times 1.75^2} = 5.364 \text{ ft. per sec.}$$

The velocity of the centre of mass is

$$v = \frac{36 \times 65 \times 2.416}{907.4 \times 2.416 + 65 \times 1.75^2} = 2.364 \text{ ft. per sec.}$$

The angular velocity is

$$\omega = -\frac{36 \times 65 \times 1.75}{907.4 \times 2.416 + 65 \times 1.75^2} = -1.712 \text{ radians per sec.}$$

CHAPTER II.

MOMENT OF INERTIA.

DETERMINATION OF MOMENT OF INERTIA. RADIUS OF GYRATION. REDUCTION OF MOMENT OF INERTIA. MOMENT OF INERTIA FOR A LINE. FOR A PLANE AREA. FOR A POINT. ELLIPSOID OF INERTIA. PRINCIPAL AXES. MINIMUM MOMENT OF INERTIA. EQUIMOMENTAL CONES. REDUCTION OF PRODUCT OF INERTIA. EQUIMOMENTAL BODIES OR SYSTEMS. MOMENTS AND PRODUCTS OF INERTIA OF BODIES.

Moment of Inertia of a Body.—We have already seen in the preceding chapter (page 172) the part played by the moment of inertia in rotary motion. In the present chapter we shall show how to determine the moment of inertia.

We may define the moment of inertia of a body with reference to any point, line or plane as *the sum of the products obtained by multiplying the mass of each element of the body by the square of its distance from that point, line or plane.*

If m is the mass of an element and r its distance from any point, line or plane, then the moment of inertia is

$$I' = \int mr^2.$$

The determination of the moment of inertia of a body is then a mere problem of integration.

We denote the moment of inertia with reference to the centre of mass, or a line or plane through the centre of mass, by I; with reference to any other point, line or plane by I'.

Radius of Gyration.—The radius of gyration of a body with reference to any point, line or plane is that distance at which, if the entire mass M of the body were concentrated in a single particle, the moment of inertia would be the same as for the body itself.

We denote the radius of gyration with reference to the centre of mass, or a line or plane through the centre of mass, by κ. For any other point, line or plane we denote it by κ'

We have then

$$I' = M\kappa'^2, \quad I = M\kappa^2,$$

or

$$\kappa'^2 = \frac{I'}{M}, \qquad \kappa = \frac{I}{M}.$$

Reduction of Moment of Inertia.—We have already found (page 173) the theorem of moment of inertia for parallel axes, viz.,

$$I' = I + Md^2,$$

or, the moment of inertia of a body with reference to any line is equal to the moment of inertia with reference to a parallel line through the centre of mass, plus the product of the mass of the body by the square of the distance between the two lines.

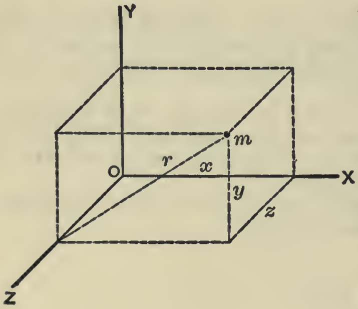
If therefore we know I and d, we can find I', or conversely, if we know I' and d, we can find I.

We have then also

$$\kappa'^2 = \kappa^2 + d^2.$$

Evidently the moment of inertia with reference to any line through the centre of mass is less than for any parallel line, and the radius of gyration with reference to any line through the centre of mass is less than with reference to any parallel line.

Moment of Inertia with Reference to a Line.—Let OZ be any line and ZOY, ZOX, any two rectangular planes passing through that line. Then for any particle of a body of mass m whose co-ordinates are x, y, z, we have the moment of inertia with reference to OZ



$$mr^2 = mx^2 + my^2.$$

Summing the moments of inertia for all the particles of the entire body, we have for the moment of inertia of the body with reference to the line OZ

$$\Sigma mr^2 = \Sigma mx^2 + \Sigma my^2.$$

But Σmr^2 is the moment of inertia of the body, I'_z, with reference to the line OZ, and $\Sigma mx^2 = I_{zy}'$, $\Sigma my^2 = I'_{zx}$, are the moments of inertia of the body with reference to the planes ZOY and ZOX. Hence

$$I'_z = I'_{zy} + I'_{zx},$$

or, *the moment of inertia of any body with reference to a line is equal to the sum of the moments of inertia for any two rectangular planes passing through that line.*

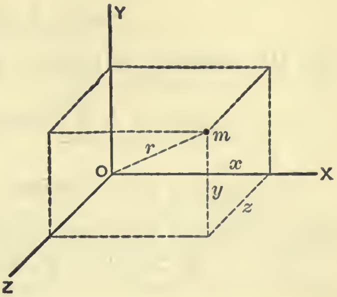
COR. For any plane area as YOX, we have

$$I'_z = I'_x + I_y',$$

or, *the moment of inertia of any plane area with reference to a line perpendicular to the plane is equal to the sum of the moments of inertia for any two rectangular lines in the plane through the foot of the perpendicular.*

Moment of Inertia with Reference to a Point.—Let O be any point,

OZ any line through that point, and YOX a plane through the point perpendicular to the line.



Then for any particle of a body of mass m whose co-ordinates are x, y, z, we have for the moment of inertia with reference to O

$$mr^2 = mx^2 + my^2 + mz^2.$$

Summing the moments of inertia for all the particles of the entire body, we have for the moment of inertia of the body with reference to O

$$\Sigma mr^2 = \Sigma mx^2 + \Sigma my^2 + \Sigma mz^2.$$

But $\Sigma mr^2 = I_0'$ is the moment of inertia of the body with reference to the point O, and $\Sigma mx^2 = I'_{zy}$, $\Sigma my^2 = I'_{zx}$, $\Sigma mz^2 = I'_{xy}$, are the moments of inertia of the body with reference to the co-ordinate planes. Hence

$$I_0' = I'_{zy} + I'_{zx} + I'_{xy}.$$

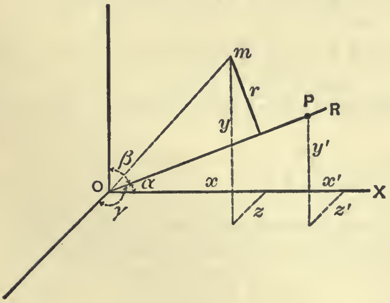
But we have just seen that $I'_{zy} + I'_{zx} = I'_z$. Hence

$$I_0' = I_z' + I'_{xy}.$$

That is, *the moment of inertia with reference to any point is equal to the sum of the moments of inertia for any three rectangular planes through that point;*

Or, *is equal to the sum of the moments of inertia for any line through the point and a plane through the point at right angles to this line.*

Ellipsoid of Inertia.—The ellipsoid of inertia gives the relations existing between the moments of inertia of a body with reference to all lines passing through any given point.



Let this point be the origin O, let m be the mass of any particle of a body whose co-ordinates are x, y, z, with reference to any assumed system of rectangular axes through O, and let OR be any line through the origin, making the angles α, β, γ with the axes. Let r be the perpendicular from m on this line, then we have

$$r^2 = x^2 + y^2 + z^2 - (x \cos \alpha + y \cos \beta + z \cos \gamma)^2.$$

We have then for the moment of inertia of m with reference to the line OR

$$mr^2 = m[x^2 + y^2 + z^2 - (x \cos \alpha + y \cos \beta + z \cos \gamma)^2].$$

Summing the moments of inertia for all the particles of the entire body, we have, since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

for the moment of inertia of the body with reference to the line OR

$$\Sigma mr^2 = \Sigma m\,[(x^2 + y^2 + z^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \\ - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2].$$

Multiplying and reducing,

$$\Sigma mr^2 = \Sigma m(y^2 + z^2)\cos^2\alpha + \Sigma m(x^2 + z_2)\cos^2\beta \\ + \Sigma m(x^2 + y^2)\cos^2\gamma - 2\Sigma myz\cos\beta\cos\gamma \\ - 2\Sigma mxz\cos\alpha\cos\gamma - 2\Sigma mxy\cos\alpha\cos\beta.$$

But (page 218)

$$\Sigma m(y^2 + z^2) = I_x', \quad \Sigma m(x^2 + z^2) = I_y', \quad \Sigma m(x^2 + y^2) = I_z',$$

are the moments of inertia of the body with reference to the axes of X, Y, and Z, and $\Sigma mr^2 = I'$ is the moment of inertia of the body with reference to the line OR.

Using this notation we have then

$$I' = I_x'\cos^2\alpha + I_y'\cos^2\beta + I_z'\cos^2\gamma - 2\Sigma myz\cos\beta\cos\gamma \\ - 2\Sigma mxz\cos\alpha\cos\gamma - 2\Sigma mxy\cos\alpha\cos\beta. \quad (1)$$

Let M be the mass of the body and κ', κ_x', κ_y', κ_z' be the radii of gyration of the body with reference to the line OR and the axes of X, Y, Z, respectively. Then

$$I' = M\kappa'^2, \quad I_x = M\kappa_x'^2, \quad I_y = M\kappa_y'^2, \quad I_z = M\kappa_z'^2,$$

and equation (1) can be written

$$\kappa'^2 = \kappa_x'^2\cos^2\alpha + \kappa_y'^2\cos^2\beta + \kappa_z'^2\cos^2\gamma - \frac{2}{M}\Sigma myz\cos\beta\cos\gamma \\ - \frac{2}{M}\Sigma mxz\cos\alpha\cos\gamma - \frac{2}{M}\Sigma mxy\cos\alpha\cos\beta. \quad (2)$$

Now suppose we lay off a distance $OP = l$ from O along the line OR and make

$$OP = l = \frac{\rho^2}{\kappa'},$$

where ρ is any arbitrary length we please, and let x', y', z' be the co-ordinates of the point P. Then whatever the assumed value of ρ, we have

$$\left.\begin{aligned} x' &= l\cos\alpha = \frac{\rho^2}{\kappa'}\cos\alpha, \quad \text{or} \quad \cos\alpha = \frac{\kappa' x'}{\rho^2}; \\ y' &= l\cos\beta = \frac{\rho^2}{\kappa'}\cos\beta, \quad \text{or} \quad \cos\beta = \frac{\kappa' y'}{\rho^2}; \\ z' &= l\cos\gamma = \frac{\rho^2}{\beta'}\cos\gamma, \quad \text{or} \quad \cos\gamma = \frac{\kappa' z'}{\rho^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

Substituting these values in (2), we obtain

$$\frac{\kappa_x'^2}{\rho^4}x'^2 + \frac{\kappa_y'^2}{\rho^4}y'^2 + \frac{\kappa_z'^2}{\rho^4}z'^2 - \frac{2}{M\rho^4}(\Sigma myz)y'z' - \frac{2}{M\rho^4}(\Sigma mxz)x'z' - \frac{2}{M\rho^4}(\Sigma mxy)x'y' = 1. \quad (4)$$

This is the equation of an ellipsoid. If we multiply by M, then since $M\rho^4 = Ml^{2'}\kappa^2 = l^2I'$, we have

$$l^2I' = I_xx'^2 + I_yy'^2 + I_zz'^2 - 2(\Sigma myz)y'z' - 2(\Sigma mxz)x'z' - 2(\Sigma mxy)x'y'. \quad (5)$$

That is, if we lay off on every line OR through the origin a distance $l = \frac{\rho^2}{\kappa'}$, where the distance ρ may have any arbitrary value, *all the points P thus determined will lie in the surface of an ellipsoid.*

This ellipsoid is called the **ellipsoid of inertia of the point** O, because *the square of the reciprocal of any one of its semi-diameters* $\left(\frac{1}{l^2} = \frac{\kappa'^2}{\rho^4}\right)$ *multiplied by the mass M of the body, is proportional to the moment of inertia* ($M\kappa'^2$) *of the body with reference to the coincident line through the point O.*

Expressions of the form Σmr_1r_2, where r_1, r_2 are the distances of an elementary mass m from two planes, are called *moments of deviation* or **products of inertia**. We adopt the latter term and denote them by D. Thus $\Sigma mxy = D_{xy}$ is the product of inertia *with reference to the XY axes.* In like manner $\Sigma myz = D_{yz}$ and $\Sigma mzx = D_{zx}$ are the products of inertia with reference to the yz and zx axes respectively.

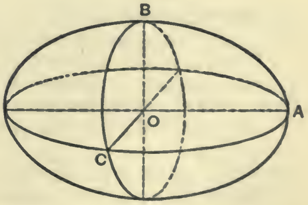
The equation of the ellipsoid of inertia for any point O can then be written

$$\frac{\kappa_x'^2}{\rho^4}x'^2 + \frac{\kappa_y'^2}{\rho^4}y'^2 + \frac{\kappa_z'^2}{\rho^4}z'^2 - \frac{2D_{xy}}{M\rho^4}x'y' - \frac{2D_{yz}}{M\rho^4}y'z' - \frac{2D_{zx}}{M\rho^4}z'x' = 1. \quad (6)$$

When the point O is the origin for any assumed set of rectangular co-ordinate axes, x', y', z' are the co-ordinates of any point of the ellipsoid for these axes; κ_x' κ_y', κ_z' are the radii of gyration of the body with reference to the axes of X, Y, Z respectively; D_{xy}, D_{yz}, D_{zx} are the respective products of inertia with reference to the xy, yz, and zx axes; M is the mass of the body, and ρ is any assumed constant length.

Principal Axes.—The point O is the centre of the ellipsoid. The axes of figure OA, OB, OC of the ellipsoid are called the **principal axes at the point** O, and the moments of inertia of the body with reference to these principal axes are called the **principal moments of inertia** *at the point O.*

These principal moments of inertia must evidently include the greatest and least of all the moments of inertia at the point O, the least corresponding to the longest semi-diameter OA, and the greatest to the shortest semi-diameter OC.



For any point O, then, there must evidently be at least one set of rectangular axes, OA, OB, OC, which are principal axes.

Now the equation of an ellipsoid referred to its centre and axes has the form

$$Ax'^2 + By'^2 + Cz'^2 = 1.$$

Comparing with equation (6), we see that the equation of the ellipsoid of inertia takes this form when

$$D_{xy} = \Sigma mxy = 0, \quad D_{yz} = \Sigma myz = 0, \quad D_{zx} = \Sigma mzx = 0, \quad (7)$$

in which case it becomes

$$\kappa_x'^2 x'^2 + \kappa_y'^2 y'^2 + \kappa_z'^2 z'^2 = \rho^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where κ_x', κ_y', κ_z' are the **principal radii of gyration** with reference to the principal axes. Equations (7) are therefore the *equations of condition for principal axes.*

If any two of these conditions, as for instance

$$D_{xy} = \Sigma mxy = 0, \qquad D_{yz} = \Sigma myz = 0,$$

are fulfilled, the equation of the ellipsoid of inertia at any point O becomes

$$\kappa_x'^2 x'^2 + \kappa_y'^2 y'^2 + \kappa_z'^2 z'^2 - \frac{2D_{zx}}{M} z'x' = \rho^4.$$

We see from this equation that for any given values of z', x', we have two equal values of y' with opposite signs. Hence the surface of the ellipsoid is symmetrical with respect to the zx plane, *and hence the axis of Y is a principal axis at the origin.*

Conversely, if a line is a principal axis at one of its points, then taking this point as origin and the line as axis of Y, the conditions

$$D_{xy} = \Sigma mxy = 0, \quad D_{yz} = \Sigma myz = 0$$

must be satisfied.

We see, moreover, that if a line is a principal axis at one of its points as O, it will not in general be a principal axis at any other of its points. For, taking the line as axis of Y and O as origin, we must have $\Sigma mxy = 0$ and $\Sigma myz = 0$. If now we take some other point on the line at a distance a from O as origin, if the line is a principal axis for this point also we must have

$$\Sigma mx(y - a) = 0, \quad \Sigma mz(y - a) = 0,$$

which can only be the case when $\Sigma mx = 0$ and $\Sigma mz = 0$, that is, *when the line passes through the centre of mass.*

Hence, *a line cannot be a principal axis at more than one of its points, unless it passes through the centre of mass; in the latter case it is a principal axis at every one of its points.*

The ellipsoid for the centre of mass is called the **central ellipsoid of inertia.**

From equations (1) and (2) we have also for the equation of the ellipsoid of inertia at a point O, referred to its principal axes,

$$I' = I_x' \cos^2 \alpha + I_y' \cos^2 \beta + I_z' \cos^2 \gamma. \quad . \quad . \quad . \quad . \quad (9)$$

$$\kappa'^2 = \kappa_x'^2 \cos^2 \alpha + \kappa_y'^2 \cos^2 \beta + \beta_z'^2 \cos^2 \gamma. \quad . \quad . \quad . \quad (10)$$

That is, *the moment of inertia of a body with reference to any line is equal to the sum of the products obtained by multiplying the principal moments of inertia for any point of the line, respectively, by the squares of the cosines of the angles which the line makes with the principal axes at that point.*

In finding the ellipsoid of inertia for a body at any point, *considerations of symmetry* are often of assistance.

Thus if a body has a plane of symmetry, then taking this plane as the yz-plane and a perpendicular to it at any point as the axis of X, we have for any given values of y, z, two equal values of x with contrary signs. Hence $\Sigma mxz = 0$ and $\Sigma mxy = 0$, whatever the position of the other two co-ordinate planes.

Therefore, *any perpendicular to a plane of symmetry is a principal axis at its point of intersection with the plane; and a perpendicular to a plane of symmetry at the centre of mass is a principal axis at every one of its points.*

If the body has two planes of symmetry at right angles to each other, then taking one as the yz-plane and the other as the zx-plane and their intersection as the axis of z, it is evident that all three products of inertia vanish, and

$$D_{xy} = \Sigma mxy = 0, \quad D_{yz} = \Sigma myz = 0, \quad D_{zx} = \Sigma mzx = 0,$$

no matter where the origin be taken on the axis of z.

Hence, *the principal axes at any point on the line of intersection of two rectangular planes of symmetry are this line of intersection and the two perpendiculars drawn to it at the point, in each plane.*

If there are three planes of symmetry at right angles to each other, their point of intersection is the centre of mass, and their lines of intersection are the principal axes at the centre of mass.

Minimum Moment of Inertia.—Let I be the moment of inertia of a body with reference to any line through the centre of mass, I' the moment of inertia with reference to any parallel line at a distance d from the first, and M the mass of the body. Then we have seen (page 218) that

$$I' = I + Md^2;$$

that is, the moment of inertia of a body with reference to any line is equal to its moment of inertia with reference to a parallel line through the centre of mass, plus the product of the mass of the body by the square of the distance between the two lines.

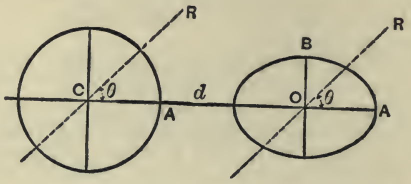
We see, then, that the moment of inertia of a body with reference to any line through the centre of mass is less than the moment of inertia with reference to any other parallel line.

Hence, *the least principal moment of inertia at the centre of mass is the least of all the moments of inertia of the body, and is equal to the mass M multiplied by the square of the reciprocal of the longest semi-diameter of the central ellipsoid.*

Discussion of the Ellipsoid of Inertia.—Let I_x, I_y, I_z be the principal moments of inertia *at the centre of mass.*

(1) Let $I_x = I_y = I_z = M\kappa^2$, where M is the mass of the body and κ is the principal radius of gyration for each principal axis. In this case we see from (9) that the central ellipsoid is a sphere, and therefore all moments of inertia at the centre of mass are equal to $M\kappa^2$ and all axes through it are principal axes. The radius of the sphere is then $\frac{1}{\kappa}$, so that the mass M multiplied by the square

of the reciprocal of any radius gives the moment of inertia for the coincident line.

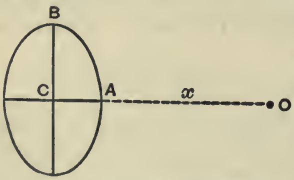


Let O be any other point at a distance $CO = d$ from the centre of mass C. Let OR be any line making the angle θ with CO. Then the moment of inertia with reference to this line is

$$I' = M\kappa^2 + Md^2 \sin^2 \theta.$$

For all lines through O perpendicular to CO the moment of inertia is then $M(\kappa^2 + d^2)$, while for the line CO the moment of inertia is $M\kappa^2$. The ellipsoid of inertia becomes then a *prolate spheroid* whose greatest principal axis is $OA = \frac{1}{\kappa}$, or the same as the radius of the sphere, while all axes through O perpendicular to OA are principal axes, and equal to $\frac{1}{\sqrt{\kappa^2 + d^2}}$.

(2) Let $I_x > I_y$ and $I_y = I_z = M\kappa_y^2$. In this case we have $M\kappa_x^2 > M\kappa_y^2$, or $\kappa_x > \kappa_y$, or $\frac{1}{\kappa_y} > \frac{1}{\kappa_x}$. The semi-diameters of the central ellipsoid along the axes of Y and Z are then both equal to $\frac{1}{\kappa_y}$, and along the axis of X the semi-diameter is $\frac{1}{\kappa_x}$. The central ellipsoid is then an *oblate spheroid* whose greatest principal axis is $CB = \frac{1}{\kappa_y}$, constant for all lines through C perpendicular to the least principal axis $CA = \frac{1}{\kappa_x}$.



There are two points on the axis CA at which the ellipsoid is a sphere of radius $\frac{1}{\kappa_x}$. At these points all moments of inertia must be equal to I_x, since I_x is unchanged by the change of point. These points can be found as follows:

Let x be the distance from C to any point O on the axis of X or on CA prolonged. If all moments are equal at this point, we must have

$$I_x = I_y + Mx^2 = I_z + Mx^2.$$

Hence

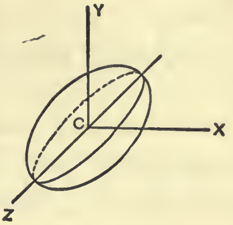
$$x = \pm \sqrt{\frac{I_x - I_y}{M}} = \pm \sqrt{\kappa_x^2 - \kappa_y^2}.$$

It is evident the ellipsoid can become a sphere at no other points.

(3) Let $I_x = I_y = M\kappa_x^2$ and $I_y > I_z$. In this case we have

$$\kappa_y > \kappa_z, \quad \text{or} \quad \frac{1}{\kappa_z} > \frac{1}{\kappa_y}.$$

The semi-diameters of the central ellipsoid along the axes of X and Y are then both equal to $\frac{1}{\kappa_y}$ and along the axis of Z the semi-diameter is $\frac{1}{\kappa_z}$. The central ellipsoid is then a *prolate spheroid* whose axis is that of Z. There is no point in this axis at which the ellipsoid becomes a sphere, because we find as before



$$= \pm \sqrt{\frac{I_z - I_y}{M}} = \pm \sqrt{\kappa_z - \kappa_y}.$$

Since $\kappa_y > \kappa_z$ we have the square root of a minus quantity.

(4) Let $I_x > I_y > I_z$. Then the central ellipsoid is one of three unequal axes at the centre of mass and cannot be a sphere at any point.

Equimomental Cones.—From equation (8) we have for the equation of the ellipsoid of inertia at any point O, referred to its centre and principal axes

$$\kappa_x'^2 x'^2 + \kappa_y'^2 y'^2 + \kappa_z'^2 z'^2 = \rho^4.$$

The equation of a sphere of radius $\frac{\rho^2}{\kappa'}$ described about O is

$$x'^2 + y'^2 + z'^2 = \frac{\rho^4}{\kappa'^2}.$$

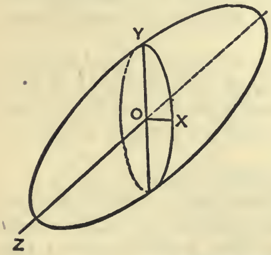
The intersections of this sphere with the ellipsoid give curves on the surface of the ellipsoid.

The radius vectors from O to every point of these curves form cones which are called the **equimomental cones.** Every straight line in the surface of these cones passing through O to the ellipsoid is a semi-diameter of the ellipsoid for which *the moment of inertia is constant.*

Combining the two equations above, we have for the equation of these cones

$$\left.\begin{array}{l} (\kappa_x'^2 - \kappa'^2)x'^2 + (\kappa_y'^2 - \kappa'^2)y'^2 + (\kappa_z'^2 - \kappa'^2)z'^2 = 0; \\ \text{or multiplying by } M, \\ (I_x' - I')x'^2 + (I_y' - I')y'^2 + (I_z' - I')z'^2 = 0. \end{array}\right\} \quad . \quad . \quad (11)$$

Let $I_x' > I_y' > I_z'$. Then $\frac{1}{\kappa_z'} > \frac{1}{\kappa_y'} > \frac{1}{\kappa_x'}$, or the semi-axis OZ of the ellipsoid is greater than the semi-axis OY, and the semi-axis OY is greater than the semi-axis OX. If, then, $\frac{1}{\kappa'}$ is less than $\frac{1}{\kappa_z'}$ and greater than $\frac{1}{\kappa_y'}$, the intersections of the sphere and ellipsoid give two tangent cones, the axis of one coincident with OZ and of the other coincident with OY. If $\frac{1}{\kappa'} = \frac{1}{\kappa_z'}$, the first of these cones becomes a straight line coincident with OZ and the other becomes a plane co-



incident with ZX. If $\frac{1}{\kappa'}$ is greater than $\frac{1}{\kappa_z'}$, there is no intersection.

If $\frac{1}{\kappa'}$ is less than $\frac{1}{\kappa_y'}$ and greater than $\frac{1}{\kappa_x'}$, the intersections of the sphere and ellipsoid give two tangent cones, the axis of one coincident with OY and of the other with OX. If $\frac{1}{\kappa'} = \frac{1}{\kappa_y'}$, the first of these becomes a plane coincident with ZX, and the other a straight line coincident with OX. If $\frac{1}{\kappa'}$ is less than $\frac{1}{\kappa_x'}$, there is no intersection.

If $\frac{1}{\kappa'} = \frac{1}{\kappa_y'}$, the cones reduce to two planes given by

$$(\kappa_x'^2 - \kappa'^2)x'^2 + (\kappa_z'^2 - \kappa'^2)z'^2 = 0.$$

These are the central circular sections or *cyclic* sections of the ellipsoid. They intersect in the axis OY and divide the ellipsoid into four wedges, whose centre lines for one pair are OZ and for the other pair OX. The first pair contains all the equimomental cones whose axes coincide with OZ or the greatest axis of the ellipsoid, the other pair contains all those whose axes coincide with OX or the least axis of the ellipsoid.

Reduction of Products of Inertia.—We have already proved (page 173) the "theorem of moment of inertia for parallel axes," viz.,

$$I' = I + Md^2;$$

that is, the moment of inertia I' of a body with reference to any line is equal to the moment of inertia I with reference to a parallel line through the centre of mass, plus the moment of inertia Md^2 of the entire mass, concentrated at the centre of mass, with reference to the original line.

We can easily prove a similar theorem for products of inertia.

Thus let D_{xy} be the product of inertia of a body with reference to any two axes X, Y through the centre of mass, D'_{xy} the product of inertia with reference to any two parallel axes, $\bar{x}$ and $\bar{y}$ the co-ordinates of the centre of mass, and M the mass of the body. Then we have the relation

$$D'_{xy} = D_{xy} + M\bar{x}\bar{y},$$

that is, *the product of inertia D'_{xy} of a body with reference to any two axes is equal to the product of inertia D_{xy} with reference to two parallel axes through the centre of mass, plus the product of inertia $M\bar{x}\bar{y}$ of the entire mass, concentrated at the centre of mass, with reference to the original axes.*

This we can call the "theorem of product of inertia for parallel axes." By means of it we can find D' for any two axes, if D for two parallel axes through the centre of mass and the co-ordinates of the centre of mass are known. Or, conversely, we can find D if D' and the co-ordinates are known.

We can easily prove this theorem as follows:

Let x', y' be the distances of any particle m from the $Y'Z'$ and $Z'X'$ planes, let $\bar{x}$, $\bar{y}$ be the distances of the centre of mass from the same planes and x, y the distances of the particle from the parallel planes YZ and ZX through the centre of mass.

Then we have $x' = \bar{x} + x$, $y' = \bar{y} + y$, and hence

$$\Sigma mx'y' = \Sigma m(\bar{x} + x)(\bar{y} + y) = \bar{x}\bar{y}\Sigma m + \Sigma mxy + \bar{x}\Sigma my + \bar{y}\Sigma mx.$$

Since the planes YZ and ZX pass through the centre of mass, we have $\Sigma my = 0$, $\Sigma mx = 0$. Hence

$$\Sigma mx'y' = \Sigma mxy + \bar{x}\bar{y}\Sigma m.$$

But $\Sigma mx'y' = D'_{xy}$, $\Sigma mxy = D_{xy}$ and $\Sigma m = M$. Therefore

$$D'_{xy} = D_{xy} + M\bar{x}\,\bar{y}.$$

Equimomental Bodies or Systems.—Two bodies or systems of bodies are said to be equimomental when their moments of inertia about all straight lines are equal each to each.

If two bodies or systems have the same centre of mass, the same mass, the same principal axes and principal moments of inertia at the centre of mass, they are equimomental.

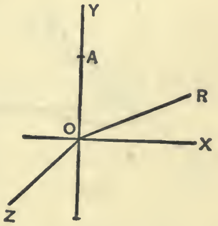
Determination of Moments and Products of Inertia.—To determine the moment of inertia of a body with reference to any line, we have simply to perform the summation Σmr^2, where m is the mass of an element and r its distance from the line.

To determine the product of inertia we have to perform the summation Σmr_1r_2, where r_1, r_2 are the distances of an element of mass m from two rectangular planes.

[(1) **Moment of Inertia of a Homogeneous Material Line.**—Let the length AB be l, and the linear density δ. Then the mass is

$$M = \delta l,$$

and the centre of mass is at the middle point O.



Let the line coincide with the axis of Y, and take the axes of X and Z through the centre of mass O. The planes XY, YZ, ZX are planes of symmetry. Hence (page 223) any three rectangular axes OX, OY, OZ through the centre of mass, of which any one, as OY, coincides with the line, are principal axes at the centre of mass.

We have then for the moment of inertia with reference to the axis of Y through the centre of mass, coinciding with the line,

$$I_y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The mass of any element of the rod is $m = \delta dy$. Hence the moment of inertia with reference to OX or OZ through the centre of mass is

$$I_x = I_z = \int_{-\frac{l}{2}}^{+\frac{l}{2}} my^2 = \int_{-\frac{l}{2}}^{+\frac{l}{2}} \delta y^2 dy = \frac{\delta l^3}{12} = M\frac{l^2}{12}. \quad . \quad . \quad (2)$$

For any line OR through the centre of mass in the plane XY, making the angle α with OX, we have from equation (9), page 222,

$$I = I_x \cos^2\alpha + I_y \sin^2\alpha = M\frac{l^2}{12}\cos^2\alpha \quad . \quad . \quad . \quad . \quad (3)$$

For any parallel line at a distance d from OR we have by the theorem of parallel axes, page 173,

$$I' = I + Md^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equimomental System.—Let the line AB be replaced by three particles of mass m_1 at the ends A and B and m_2 at the centre of mass O. Then we have

$$2m_1 + m_2 = M,$$

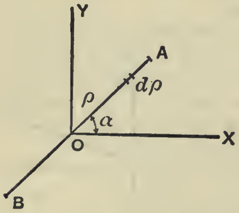
and for a line through B parallel to OX, since in this case $d = \frac{l}{2}$,

$$m_1 l^2 + m_2 \left(\frac{l}{2}\right)^2 = M\frac{l^2}{12} + M\left(\frac{l}{2}\right)^2.$$

From these two equations we obtain

$$m_1 = \frac{1}{6}M, \qquad m_2 = \frac{2}{3}M.$$

Hence, *the moment of inertia of a homogeneous material line with reference to any line whatever is the same as for a system consisting of a particle of one-sixth the mass at each end and a particle of two thirds the mass at the centre of mass.*



Product of Inertia.—Take the axes of X and Y through the centre of mass O, and let the line AB be in the plane XY and make the angle α with the axis of X. Let ρ be the distance of any element from O, and δ the linear density. Then the mass of an element is $m = \delta d\rho$ and its co-ordinates are $x = \rho\cos\alpha$, $y = \rho\sin\alpha$. The mass of the line is $M = \delta l$.

We have then for the product of inertia for two rectangular axes X, Y through the centre of mass, the line being in the plane XY and making the angle α with the axis of X,

$$D_{xy} = \int_{-\frac{l}{2}}^{+\frac{l}{2}} mxy = \int_{-\frac{l}{2}}^{+\frac{l}{2}} \delta \sin\alpha \cos\alpha\rho^2 d\rho$$

$$= \frac{\delta l^3}{12}\sin\alpha\cos\alpha = M\frac{l^2}{12}\sin\alpha\cos\alpha \quad . \quad . \quad . \quad . \quad (5)$$

For any pair of parallel axes X', Y' we have by the theorem of parallel axes (page 173)

$$D'_{xy} = D_{xy} + M\bar{x}\,\bar{y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

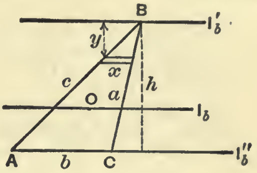
where $\bar{x}$, $\bar{y}$ are the co-ordinates of the centre of mass O with reference to X', Y'.

If the line coincides with the axis of X or Y we have

$$D_{xy} = 0, \quad D'_{xy} = M\bar{x}\,\bar{y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We see, then, that *the product of inertia of a homogeneous material line with reference to any pair of rectangular axes is, like the moment of inertia, the same as for a system consisting of a particle of one sixth the mass at each end and a particle of two thirds the mass at the centre of mass.*

[(2) **Moment of Inertia of a Homogeneous Material Triangle.**—Let A, B, C represent the angles of a triangle; a, b, c the sides opposite respectively; h the altitude for any side b, and δ the surface density. Then the mass of the triangle is



$$M = \frac{\delta b h}{2}.$$

Take an elementary strip parallel to the side b at a distance y from the vertex B, and let x be the length of this strip and dy its thickness. Then we have

$$x : y :: b : h, \quad \text{or} \quad x = \frac{b}{h} y.$$

The area of the strip is $x dy = \frac{b}{h} y \, dy$, and its mass is $m = \frac{\delta b}{h} y \, dy$. We can consider this strip as a material line. Its moment of inertia with reference to the coincident line is then zero. With reference to the parallel line through the vertex B it is then my^2. The moment of inertia of the triangle with reference to this line is then

$$I_b' = \int_0^h my^2 = \int_0^h \frac{\delta b}{h} y^3 dy = \frac{\delta b h^3}{4} = M\frac{h^2}{2}. \quad . \quad . \quad . \quad (1)$$

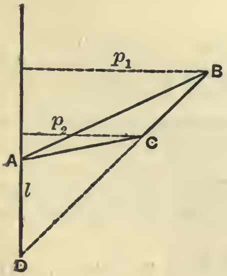
For the parallel axis through the centre of mass O we have by the theorem of parallel axes (page 173)

$$I_b = I_b' - M\left(\frac{2}{3}h\right)^2 = M\frac{h^2}{18}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the axis coinciding with the base b we have

$$I_b'' = I_b + M\left(\frac{h}{3}\right)^2 = M\frac{h^2}{6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now take any axis AD through the vertex A in the plane of the triangle. Let p_1, p_2 be the perpendiculars from B and C on AD. Produce the side DC to intersection D with AD, and let the distance $AD = l$.



Let M_1 be the mass of the triangle ADB, so that $M_1 = \frac{\delta l p_1}{2}$. The moment of inertia of this triangle with reference to the line AD coinciding with the base is, from (3),

$$I_1' = M_1\frac{p_1^2}{6} = \frac{\delta l p_1^3}{12}.$$

Let M_2 be the area of the triangle ADC, so that $M_2 = \frac{\delta l p_2}{2}$. The moment of inertia of this triangle with reference to AD coinciding with its base is, from (3),

$$I_2' = M_2\frac{p_2^2}{6} = \frac{\delta l p_2^3}{12}.$$

Hence the moment of inertia of the given triangle ABC with reference to AD is

$$I' = I_1' - I_2' = \frac{\delta l}{12}(p_1^3 - p_2^3) = \frac{\delta l}{2}(p_1 - p_2)\,.\,\frac{1}{6}(p_1^2 + p_1p_2 + p_2^2).$$

But $\frac{\delta l}{2}(p_1 - p_2)$ is the mass M of the triangle ABC, and

$$\frac{1}{6}(p_1^2 + p_1p_2 + p_2^2) = \frac{1}{3}\left[\left(\frac{p_1}{2}\right)^2 + \left(\frac{p_2}{2}\right)^2 + \left(\frac{p_1 + p_2}{2}\right)^2\right].$$

Hence

$$I' = \frac{M}{3}\left[\left(\frac{p_1}{2}\right)^2 + \left(\frac{p_2}{2}\right)^2 + \left(\frac{p_1 + p_2}{2}\right)^2\right]. \quad . \quad . \quad . \quad . \quad (4)$$

That is, *the moment of inertia of a homogeneous triangle about any line is the same as for a system consisting of a particle of one third the mass of the triangle placed at the middle point of each side.*

If the axis through A is at right angles to the side b we have then

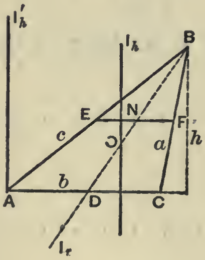
$$I_h' = \frac{M}{3}\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2\tan A}\right)^2 + \left(\frac{b}{2} + \frac{h}{2\tan A}\right)^2\right],$$

or

$$I_h' = \frac{M}{6}\left(b^2 + \frac{bh}{\tan A} + \frac{h^2}{\tan^2 A}\right) \quad . \quad (5)$$

The distance from this axis to the centre of mass O is

$$\frac{1}{3}\left(b + \frac{h}{\tan A}\right).$$



Therefore by the theorem of parallel axes (page 173) the moment of inertia with reference to a line in the plane of the triangle at right angles to the side b through the centre of mass is

$$I_h = I_h' - M\,.\,\frac{1}{9}\left(b + \frac{h}{\tan A}\right)^2 = \frac{M}{18}\left(b^2 - \frac{bh}{\tan A} + \frac{h^2}{\tan^2 A}\right). \quad (6)$$

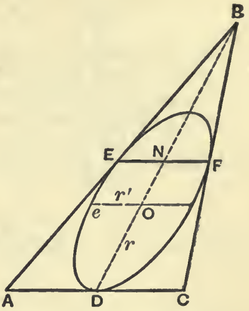
Now the plane of the triangle is a plane of symmetry, and therefore (page 223) the line through the centre of mass at right angles to this plane is a principal axis at the centre of mass.

We have then for the moment of inertia with reference to a line through the centre of mass at right angles to the plane of the triangle

$$I_z = I_b + I_h = \frac{M}{18}\left(h^2 + b^2 - \frac{bh}{\tan A} + \frac{h^2}{\tan^2 A}\right) = \frac{M}{36}(a^2 + b^2 + c^2). \quad (7)$$

Draw the line BD from the apex B to the middle point D of the base b, and let this line make the angle ω with the base. This line is a line of symmetry, passes through the centre of mass O, and bisects every line parallel to the base. Then if E, F are the middle points of the other sides, we have $EF = \frac{b}{2}$ and $EN = FN = \frac{b}{4}$. Taking $\frac{1}{3}M$ concentrated at D, E and F, we have for the moment of inertia with reference to the line BD

$$I_r = \frac{2}{3}M\left(\frac{b}{4}\sin\omega\right)^2 = \frac{Mb^2}{24}\sin^2\omega. \qquad (8)$$



Now suppose an ellipse inscribed in the triangle ABC touching two of the sides AB, BC in their middle points E, F. Then it touches the third side AC in its middle point D. Since EF is parallel to AC, the tangent at D, the straight line BD is a line of symmetry and passes through the middle point N of EF and the centre of mass O, which is also the centre of the ellipse.

Let $OD = r$, and let $Oe = r'$ be the semi-conjugate diameter, parallel to AC, and ω the angle between r and r'. Then, since the area of an ellipse is equal to $\pi \times$ rectangle of the semi-axes, we have for the area Δ' of the ellipse

$$\Delta' = \pi r r' \sin\omega.$$

Now $ON = \frac{1}{2}r$, and hence from the equation to the ellipse $EN^2 = \frac{3}{4}r'^2$.

If then we take $\frac{1}{3}M$ concentrated at D, E and F, we have for the moment of inertia of the triangle with reference to the line BD

$$I_r = \frac{2}{3}M \cdot \frac{3}{4}r'^2 \sin^2\omega = \frac{M}{2} \cdot \frac{\Delta'^2}{\pi^2 r^2}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We see then that the moments of inertia with reference to OD, OE, OF are inversely proportional to OD^2, OE^2, OF^2. This is also the case for the ellipse of inertia. The ellipse of inertia coincides then with the inscribed ellipse at the points D, E, F, and also at the opposite ends of the diameters through these points. But two conics cannot cut each other in six points unless they are identical. Hence the inscribed ellipse *is an ellipse of inertia.* Let

$$l = \frac{\rho^2}{\kappa}$$

be any semi-diameter of this ellipse, where ρ is any arbitrary distance and κ is the radius of gyration for the axis coinciding with l.

Then the square of the reciprocal of any semi-dameter $\left(\frac{1}{l^2} = \frac{\kappa^2}{\rho^4}\right)$ multiplied by the mass M is proportional to the moment of inertia $M\kappa^2$ with reference to the coincident line through the centre of mass O.

We can take ρ any arbitrary length. Thus in the present case we have

$$\frac{M}{r^2} = \frac{I_r}{\rho^4}, \text{ or from (9) } \rho^4 = \frac{\Delta'^2}{2\pi^2}.$$

For this value of ρ the ellipse of inertia is tangent at the middle points D, E, F of the sides.

If we take $\frac{M}{(2r)^2} = \frac{I_r}{\rho^4}$ we have $\rho^4 = \frac{2\Delta'^2}{\pi^2}$.

For this value of ρ the ellipse circumscribing the triangle and having its centre at the centre of mass O is the ellipse of inertia.

Let κ_r, $\kappa_{r'}$ be the radii of gyration for the axes r and r', and κ_x, κ_y be the radii of gyration for the principal axes of the ellipse. From (8) we have $\kappa_r = \frac{b \sin \omega}{2\sqrt{6}}$, and from (2) $\kappa_{r'} = \frac{h}{3\sqrt{2}}$. We have then

$$r = \frac{\rho^2}{\kappa_r} = \frac{2\sqrt{6}\rho^2}{b \sin \omega}, \qquad r' = \frac{\rho^2}{\kappa_{r'}} = \frac{3\sqrt{2}\rho^2}{h}.$$

The lengths of the principal semi-axes are $\frac{\rho^2}{\kappa_x}$, $\frac{\rho^2}{\kappa_y}$. Now the parallelogram upon two conjugate semi-diameters is equal to the rectangle of the principal semi-axes. Hence

$$rr' \sin \omega = \frac{\rho^4}{\kappa_x \kappa_y}, \quad \text{or} \quad \frac{1}{\kappa_x \kappa_y} = \frac{6\sqrt{12}}{bh}.$$

We have also from (7)

$$\kappa_x^2 + \kappa_y^2 = \frac{1}{36}\left(a^2 + b^2 + c^2\right).$$

Solving these two equations, we obtain for the principal axes at the centre of mass, if $\Delta = \frac{bh}{2}$ = the area of the triangle,

$$\kappa_x^2 = \frac{1}{72}\left[(a^2 + b^2 + c^2) + \sqrt{(a^2 + b^2 + c^2)^2 - 48\Delta^2}\right]; \quad . \quad (10)$$

$$\kappa_y^2 = \frac{1}{72}\left[(a^2 + b^2 + c^2) - \sqrt{(a^2 + b^2 + c^2)^2 - 48\Delta^2}\right] \quad . \quad . \quad (11)$$

We have then for the angle θ_x which the principal axis of X makes with the base b

$$\kappa_x^2 \cos^2 \theta_x + \kappa_y^2 \sin^2 \theta_x = \frac{h^2}{18}, \quad \text{or} \quad \cos^2 \theta_x = \frac{\frac{h^2}{18} - \kappa_y^2}{\kappa_x^2 - \kappa_y^2}. \quad (12)$$

Equation (12) locates the principal axis of X with reference to the base b.

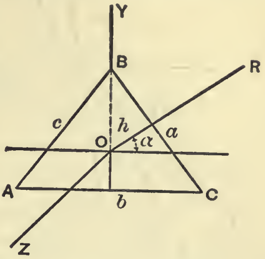
For any axis in the plane of the triangle through the centre of mass, making the angle α with the axis of X, we have then

$$\kappa_x^2 \cos^2 \alpha + \kappa_y^2 \sin^2 \alpha = \kappa^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

COR. 1. For an isosceles triangle we have $a = c$, and $h^2 = c^2 - \frac{b^2}{4}$.

Hence from (10), (11) and (12), $\kappa_x^2 = \frac{h^2}{18}$, $\kappa_y^2 = \frac{b^2}{24}$, and $\cos^2\theta_x = 1$ or $\theta_x = 0$. Hence the principal axes at the centre of mass in the plane of the triangle are parallel and perpendicular to the base b, and we have for the moment of inertia I with reference to any line OR in the plane of the triangle through the centre of mass O, making the angle α with the base b,

$$I = \frac{Mh^2}{18}\cos^2\alpha + \frac{Mb^2}{24}\sin^2\alpha. \qquad (14)$$



For the moment of inertia with reference to a line OZ through the centre of mass O at right angles to the plane of the triangle, we have from (7)

$$I_z = \frac{M}{18}\left(h^2 + \frac{3}{4}b^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

For any line through the centre of mass making the angles α, β, γ with the principal axes, we have

$$I = \frac{Mh^2}{18}\cos^2\alpha + \frac{Mb^2}{18}\cos^2\beta + \frac{M}{18}\left(h^2 + \frac{3}{4}b^2\right)\cos^2\gamma. \quad . \qquad (16)$$

COR. 2. For an equilateral triangle we have for any line in the plane of the triangle through the centre of mass

$$I = \frac{Mb^2}{24},$$

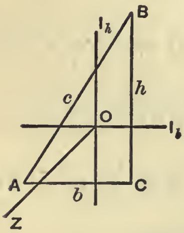
no matter what the angle with the base b.

For the polar moment of inertia

$$I_z = \frac{Mb^2}{12}.$$

For any line through the centre of mass

$$I = \frac{Mb^2}{24}\cos^2\alpha + \frac{Mb^2}{24}\cos^2\beta + \frac{Mb^2}{12}\cos^2\gamma.$$



COR. 3. For a right-angled triangle we have $a = h$, $a^2 + b^2 = c^2$.

Hence from (10), (11) and (12)

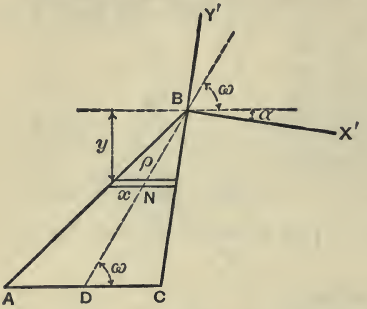
$$\kappa_x^2 = \frac{c^2 + \sqrt{c^4 - 3h^2b^2}}{36}, \quad \kappa_y^2 = \frac{c^2 - \sqrt{c^4 - 3h^2b^2}}{36},$$

$$\cos^2\theta_x = \frac{\frac{h^2}{18} - \kappa_y^2}{\kappa_x^2 - \kappa_y^2}.$$

Also, we have

$$I = M\kappa_x^2\cos^2\alpha + M\kappa_y^2\sin^2\alpha, \quad I_z = \frac{Mc^2}{18}, \quad I_b = \frac{Mh^2}{18}, \quad I_h = \frac{Mb^2}{18}.$$

Product of Inertia.—Take as before an elementary strip parallel to the side b at a distance y from the vertex B. Then we have as before the mass of this strip $m = \frac{\delta b}{h} y\,dy$, and its length $x = \frac{b}{h} y$.



From page 228 we have for the product of inertia of this strip for two rectangular axes X, Y through its centre of mass, the slice being in the plane XY and making an angle α with the axis of X,

$$d_{xy} = \frac{mx^2}{12} \sin\alpha \cos\alpha = \frac{\delta b^3}{12h^3} \sin\alpha \cos\alpha \,.\, y^3 dy.$$

Let BD be a line of symmetry passing through the vertex B and the middle D of the side b, and therefore passing through the centre of mass N of the slice. Let ρ be the distance BN, and let the line BN make the angle ω with the side b.

Then we have $\rho \sin\omega = y$, or $\rho = \frac{y}{\sin\omega}$, and the co-ordinates of the centre of mass N for two parallel rectangular axes X', Y' through the vertex B are

$$\bar{x} = \rho\cos(\omega + \alpha) = -\frac{y}{\sin\omega}\cos(\omega + \alpha);$$

$$\bar{y} = \rho\sin(\omega + \alpha) = -\frac{y}{\sin\omega}\sin(\omega + \alpha).$$

The product of inertia of the slice with reference to these axes is then

$$d'_{xy} = d_{xy} + m\overline{xy}$$

$$= \frac{\delta b^3}{12h^3} \sin\alpha\cos\alpha\, y^3 dy + \frac{\delta b}{h\sin^2\omega} \sin(\omega + \alpha)\cos(\omega + \alpha) y^3 dy.$$

The product of inertia of the triangle with reference to the two rectangular axes X', Y' through the vertex B in the plane of the triangle, if the side b makes the angle α with the axis of X', is then

$$D'_{xy} = \int_{-h}^{0} \frac{\delta b^3}{12h^3} \sin\alpha\cos\alpha\, y^3 dy + \frac{\delta b}{h\sin^2\omega} \sin(\omega + \alpha)\cos(\omega + \alpha)\, y^3 dy$$

$$= \frac{\delta b^3 h}{48} \sin\alpha\cos\alpha + \frac{\delta b^3}{4\sin^2\omega} \sin(\omega + \alpha)\cos(\omega + \alpha)$$

$$= \frac{M}{2}\left[\frac{b^2}{12} \sin\alpha\cos\alpha + \frac{h^2}{\sin^2\omega} \sin(\omega + \alpha)\cos(\omega + \alpha)\right]. \quad . \quad . \quad (17)$$

The co-ordinates of the centre of mass of the triangle are

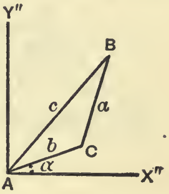
$$\bar{x} = \frac{2h}{3\sin\omega}\cos(\omega + \alpha), \qquad \bar{y} = \frac{2h}{3\sin\omega}\sin(\omega + \alpha).$$

The product of inertia of the triangle with reference to two rectangular axes X, Y through the centre of mass in the plane of the triangle, if the side b makes the angle α with the axis of X, is then

$$D_{xy} = D'_{xy} - M\bar{x}\,\bar{y} = \frac{M}{6}\left[\frac{b^2}{4}\sin\alpha\cos\alpha + \frac{h^2}{3\sin^2\omega}\sin(\omega+\alpha)\cos(\omega+\alpha)\right]. \qquad (18)$$

This is the same as for a particle of one third the mass M at the middle point of each side.

We see, then, that *the product of inertia of a homogeneous triangle with reference to any pair of rectangular axes is, like the moment of inertia, the same as for a system consisting of one third the mass of the triangle placed at the middle point of each side.*



We have then for the two axes X'', Y'' through the vertex A in the plane of the triangle, if the side b makes the angle α with the axis of X'',

$$D''_{xy} = \frac{M}{3}\left[\frac{b^2}{4}\sin\alpha\cos\alpha + \frac{c^2}{4}\sin(A+\alpha)\cos(A+\alpha) + \frac{[b\sin\alpha + c\sin(A+\alpha)][b\cos\alpha + c\cos(A+\alpha)]}{4}\right];$$

or reducing,

$$D''_{xy} = \frac{M}{6}\left[b^2\sin\alpha\cos\alpha + c^2\sin(A+\alpha)\cos(A+\alpha) + \frac{bc}{2}\sin(2\alpha+A)\right].$$

The co-ordinates of the centre of mass are

$$\bar{x} = \frac{b\cos\alpha + c\cos(A+\alpha)}{3}, \qquad \bar{y} = \frac{b\sin\alpha + c\sin(A+\alpha)}{3}.$$

Hence we have also

$$D_{xy} = D'_{xy} - M\bar{x}\,\bar{y} = \frac{M}{18}\left[b^2\sin\alpha\cos\alpha + c^2\sin(A+\alpha)\cos(A+\alpha) - \frac{bc}{2}\sin(2\alpha+A)\right]. \qquad (19)$$

COR. If $\alpha = 0$, we have, from (18) and (19),

$$D_{xy} = \frac{Mh^2}{18\tan\omega} = \frac{Mh}{18}\left[\frac{h}{\tan A} - \frac{b}{2}\right].$$

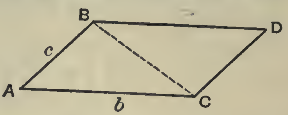
For an isosceles triangle $\omega = 90$ and therefore $\sin(\omega \times \alpha) = \cos\alpha$ and $\cos(\omega + \alpha) = -\sin\alpha$, and we have from (18)

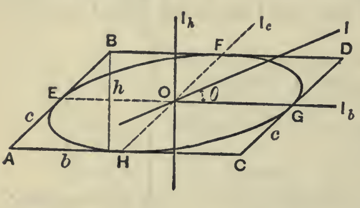
$$D_{xy} = \frac{M}{6}\left[\left(\frac{b^2}{4} - \frac{h^2}{3}\right)\sin\alpha\cos\alpha\right].$$

If in this $\alpha = 0$, we have $D_{xy} = 0$.

For an equilateral triangle $\omega = 90$ and $h^2 = \frac{3}{4}b^2$, hence $D_{xy} = 0$.

(3) Moment of Inertia of a Homogeneous Material Parallelogram.—Since a parallelogram is composed of two equal triangles ABC and BDC, and since the moment of inertia of a triangle with reference to any line is the same as for one third the mass at the middle of each side, it is evident that *the moment of inertia of a parallelogram with reference to any line is the same as for a particle of one sixth the mass of the parallelogram at the middle point of each side, and a particle of one third the mass of the parallelogram at its centre of mass.*





If A is the acute angle and the adjacent sides are $AC = b$ and $AB = c$, we have then for the line EG through the centre of mass O in the plane of the parallelogram, parallel to the side b,

$$I_b = \frac{1}{3}M\left(\frac{c}{2}\sin A\right)^2 = \frac{M}{12}c^2\sin^2 A, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $c \sin A$ is the altitude h of the parallelogram for the base b.

The moment of inertia with reference to the line through the centre of mass O in the plane of the parallelogram, at right angles to the side b, is

$$I_h = \frac{M}{3}\left(\frac{c}{2}\cos A\right)^2 + \frac{M}{3}\left(\frac{b}{2}\right)^2 = \frac{M}{12}\left(b^2 + c^2\cos^2 A\right). \quad . \quad . \quad (2)$$

The plane of the parallelogram is a plane of symmetry, and therefore (page 223) the line through the centre of mass at right angles to the plane is a principal axis at the centre of mass.

We have then for the moment of inertia with reference to a line through the centre of mass at right angles to the plane of the parallelogram

$$I_z = I_b + I_h = \frac{M}{12}\left(b^2 + c^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For the moment of inertia with reference to any line in the plane of the parallelogram through the centre of mass O, making the angle θ with the side b, we have

$$I = \frac{M}{3}\left(\frac{c}{2}\sin(A-\theta)\right)^2 + \frac{M}{3}\left(\frac{b}{2}\sin\theta\right)^2$$

$$= \frac{M}{12}\left[c^2\sin^2(A-\theta) + b^2\sin^2\theta\right]. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The moment of inertia with reference to a line HF in the plane of the parallelogram through the centre of mass O, parallel to the side c, is

$$I_c = \frac{M}{3}\left(\frac{b}{2}\sin A\right)^2 = \frac{M}{12}b^2\sin^2 A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now suppose an ellipse inscribed in the parallelogram touching the sides at the middle points E, F, G, H. The area $\varDelta$ of the parallelogram is

$$\varDelta = bc\sin A; \quad \text{hence,} \quad c\sin A = \frac{\varDelta}{b}.$$

We have then, from (1),

$$I_b = \frac{M}{12} \cdot \frac{\Delta^2}{b^2} = \frac{M}{3} \cdot \frac{\Delta^2}{16\left(\frac{b}{2}\right)^2}.$$

We see then that the moments of inertia with reference to OE, OF are inversely proportional to OE^2, OF^2. The inscribed ellipse is therefore an ellipse of inertia. Let

$$l = \frac{\rho^2}{\kappa}$$

be any semi-diameter, where ρ is any arbitrary distance and κ is the radius of gyration for the coincident line. Then the square of the reciprocal of any semi-diameter $\left(\frac{1}{l^2} = \frac{\kappa^2}{\rho^4}\right)$ multiplied by the mass M is proportional to the moment of inertia $M\kappa^2$ with reference to the coincident line through the centre of mass O.

We have then in the present case

$$\frac{M}{\left(\frac{b}{2}\right)^2} = \frac{I_b}{\rho^4}, \quad \text{or} \quad \rho^4 = \frac{\Delta^2}{48}.$$

Let κ_b, κ_c be the radii of gyration for the axes EG and HF, parallel respectively to the sides a and c, and let κ_x and κ_y be the radii of gyration for the principal axes of the ellipse. From (1) and (5) we have

$$\kappa_b = \frac{c \sin A}{2\sqrt{3}}, \quad \kappa_c = \frac{b \sin A}{2\sqrt{3}}.$$

We have then

$$OE = \frac{\rho^2}{\kappa_b} = \frac{2\sqrt{3}\rho^2}{c \sin A}, \quad OF = \frac{\rho^2}{\kappa_c} = \frac{2\sqrt{3}\rho^2}{b \sin A}.$$

The lengths of the principal semi-axes are $\frac{\rho^2}{\kappa_x}$, $\frac{\rho^2}{\kappa_y}$. Now the parallelogram upon two conjugate semi-diameters is equal to the rectangle of the principal semi-axes. Hence

$$OE \cdot OF \sin A = \frac{\rho^4}{\kappa_x \kappa_y}, \quad \text{or} \quad \frac{1}{\kappa_x \kappa_y} = \frac{12}{bc \sin A} = \frac{12}{\Delta}.$$

We have also, from (3),

$$\kappa_x^2 + \kappa_y^2 = \frac{b^2 + c^2}{12}.$$

Solving these two equations, we obtain for the principal axes, if Δ is the area of the parallelogram $= bc \sin A$,

$$\kappa_x^2 = \frac{1}{24}\left[b^2 + c^2 - \sqrt{(b^2 + c^2)^2 - 4\Delta^2}\right]; \quad . \quad . \quad . \quad (6)$$

$$\kappa_y^2 = \frac{1}{24}\left[b^2 + c^2 + \sqrt{(b^2 + c^2)^2 - 4\Delta^2}\right]. \quad . \quad . \quad . \quad (7)$$

We have then for the angle θ_x which the principal axis of X makes with the side b

$$\kappa_x^2 \cos^2 \theta_x + \kappa_y^2 \sin^2 \theta_x = \frac{c^2 \sin^2 A}{12}, \quad \text{or} \quad \cos^2 \theta_x = \frac{\frac{c^2 \sin^2 A}{12} - \kappa_y^2}{\kappa_x^2 - \kappa_y^2}. \quad (8)$$

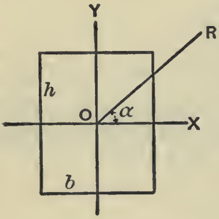
Equation (8) locates the principal axis of X with reference to the side b.

For any line in the plane of the parallelogram through the centre of mass, making the angle α with the axis of X, we have

$$\kappa_x^2 \cos^2 \alpha + \kappa_y^2 \sin^2 \alpha = \kappa^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

COR. For a rectangle $c = h$, $c \sin A = h$, $A = 90°$ and $\varDelta = bh$. Hence $\kappa_x^2 = \frac{h^2}{12}$, $\kappa_y^2 = \frac{b^2}{12}$, $\theta_x = 0$, and therefore the principal axes at the centre of mass are parallel to b and h.

We have then for any line OR in the plane of the rectangle through the centre of mass, making the angle α with the base,



$$I = \frac{Mh^2}{12} \cos^2 \alpha + \frac{Mb^2}{12} \sin^2 \alpha. \quad . \quad (10)$$

Hence

$$I_x = \frac{Mh^2}{12}, \quad I_y = \frac{Mb^2}{12}, \quad . \quad . \quad . \quad . \quad (11)$$

and for the polar moment of inertia

$$I_z = \frac{M(h^2 + b^2)}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

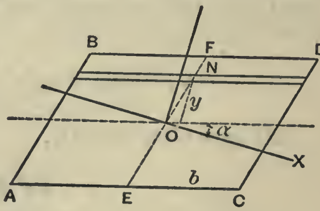
For any line through the centre of mass making the angles α, β, γ, with the principal axes

$$I = \frac{Mh^2}{12} \cos^2 \alpha + \frac{Mb^2}{12} \cos^2 \beta + \frac{M(h^2 + b^2)}{12} \cos^2 \gamma. \quad . \quad . \quad (13)$$

[**Product of Inertia.**—Take an elementary strip parallel to the side b at a distance y from the centre of mass O.

The mass of this strip is $m = \delta b dy$, and its length is b.

From page 228 we have for the product of inertia of this strip for two rectangular axes X, Y, through its centre of mass, the strip being in the plane XY and making an angle α with the axis of X,



$$d_{xy} = \frac{mb^2}{12} \sin \alpha \cos \alpha = \frac{\delta b^3}{12} \sin \alpha \cos \alpha . dy.$$

Let EF be a line of symmetry passing through the middle points E and F of the two opposite sides AC, BD, and therefore passing through the centre of mass N of the strip. Let ρ be the distance ON, making the angle A with the side b. Then $\rho \sin A = y$, or $\rho = \frac{y}{\sin A}$, and the co-ordinates of the centre of mass N for two parallel rectangular axes through the centre of mass O are

$$\bar{x} = \rho \cos (A + \alpha) = \frac{y}{\sin A} \cos (A + \alpha),$$

$$\bar{y} = \rho \sin (A + \alpha) = \frac{y}{\sin A} \sin (A + \alpha).$$

The product of inertia of the strip with reference to these axes is then

$$d'_{xy} = d_{xy} + m\bar{x}\bar{y} = \frac{\delta b^3}{12} \sin \alpha \cos \alpha \,.\, dy + \frac{\delta b}{\sin^2 A} \sin (A+\alpha) \cos (A+\alpha) y^2 dy.$$

The product of inertia of the parallelogram with reference to two rectangular axes X, Y, through the centre of mass O in the plane of the parallelogram, if the side b makes the angle α with the axis of X, is

$$D_{xy} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{\delta b^3}{12} \sin \alpha \cos \alpha \,.\, dy + \frac{\delta b}{\sin^2 A} \sin (A + \alpha) \cos (A + \alpha) y^2 dy,$$

or, since $\delta bh = M =$ the mass of the parallelogram,

$$D_{xy} = \frac{M}{12} \left[b^2 \sin \alpha \cos \alpha + \frac{h^2}{\sin^2 A} \sin (A + \alpha) \cos (A + \alpha) \right]. \quad (14)$$

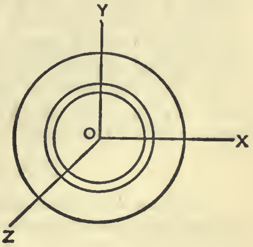
COR. 1. For a rectangle $A = 90°$, $\sin (A + \alpha) = \cos \alpha$, $\cos (A + \alpha) = - \sin \alpha$, and hence

$$D_{xy} = \frac{M}{12} (b^2 - h^2) \sin \alpha \cos \alpha \,. \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

For a square $b = h$ and $D_{xy} = 0$.

COR. 2. We see from (14) that *the product of inertia of a parallelogram for any pair of rectangular axes is, like the moment of inertia, the same as for a particle of* $\frac{1}{6}M$ *at the middle point of each side and* $\frac{1}{3}M$ *at the centre of mass.*

[(4) **Moment of Inertia of a Homogeneous Circular Disk.**—Let r be the radius and ρ the radius of any elementary circular strip of thickness $d\rho$. Then the area of this element is $2\pi\rho d\rho$, and if δ is the surface density, its mass is $2\pi\delta\rho d\rho$.



The moment of inertia of this strip with reference to the axis OZ through the centre of mass at right angles to the plane of the disk is then

$$2\pi\delta\rho^3 d\rho,$$

and the moment of inertia of the disk with reference to this axis is therefore

$$I_z = \int_0^r 2\pi\delta\rho^3 d\rho = \frac{\pi\delta r^4}{2},$$

or, since the mass of the disk is $M = \pi\delta r^2$, we have

$$I_z = M\frac{r^2}{2}.$$

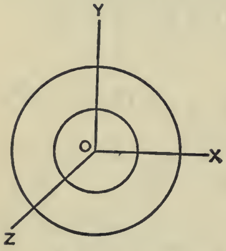
The moment of inertia with reference to any line OX or OY through the centre of mass in the plane of the disk is evidently the same for all lines. We have then

$$I_x = I_y \quad \text{and} \quad I_x + I_y = I_z.$$

Hence

$$I_x = I_y = \frac{I_z}{2} = M\frac{r^2}{4}.$$

The moment of inertia of a homogeneous circular disk is then *the same as for a particle of one eighth the mass of the disk at the extremities of any two rectangular diameters and a particle of one half the mass of the disk at its centre.*



For a hollow disk let r_1 be the outside and r_2 the inside radius. Then we have

$$I_z = \int_{r_2}^{r_1} 2\pi\delta\rho^3 d\rho = \frac{\pi\delta}{2}(r_1^4 - r_2^4),$$

or the moment of inertia of the whole disk minus that of the hollow space. But

$$r_1^4 - r_2^4 = (r_1^2 + r_2^2)(r_1^2 - r_2^2) \quad \text{and} \quad \pi\delta(r_1^2 - r_2^2) = M.$$

Hence we have

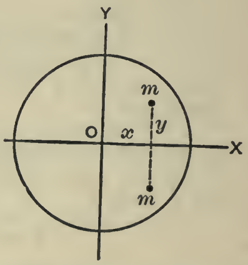
$$I_z = \frac{1}{2}M(r_1^2 + r_2^2) \quad \text{and} \quad I_x = I_y = \frac{1}{4}M(r_1^2 + r_2^2).$$

For a circular ring we have $r_1 = r_2 = r$, and hence

$$I_z = Mr^2, \quad I_x = I_y = \frac{1}{2}Mr^2.$$

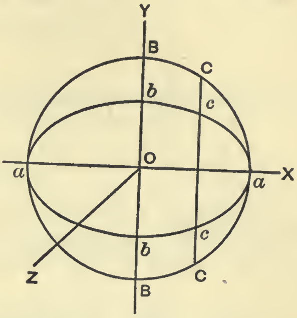
Product of Inertia.—Every diameter through the centre of mass is a line of symmetry. If then we take any pair of rectangular axes X, Y, through the centre of mass O in the plane of the circle, we have for any value of x two equal particles m, m, with equal and opposite ordinates $+y$, $-y$. The product of inertia Σmxy for these two particles is then zero. The same holds for every pair of particles. Hence the product of inertia for a homogeneous circular disk for any pair of rectangular axes through the centre of mass in the plane of the disk is zero. We have then

$$D'_{xy} = M\bar{x}\bar{y}.$$



Hence, *the product of inertia of a homogeneous circular disk for any pair of rectangular axes is, like the moment of inertia, the same as for a particle of one eighth the mass of the disk at the extremities of any two rectangular diameters and a particle of one half the mass of the disk at its centre of mass.*

(5) Moment of Inertia of a Homogeneous Ellipse.—Let the semi-transverse axis be a and the semi-conjugate axis be b. Let a circle be described about the ellipse, so that its radius Oa is equal to the semi-transverse axis a.



Then we have for the ratio of the mass of any element cc of the ellipse to that of the corresponding element CC of the circle

$$\frac{cc}{CC} = \frac{bb}{BB} = \frac{b}{a}.$$

Hence the moment of inertia of the ellipse with reference to the principal axis OY is $\frac{b}{a}$ times that of the circle. In the same way the moment of inertia of the ellipse with reference to the principal axis OX is $\frac{a}{b}$ times that of the circle.

We have then from the preceding article, since $\delta\pi a^2$ is the mass of the circle and $\delta\pi ab = M$ is the mass of the ellipse,

$$I_y = \frac{\delta\pi ba^4}{4a} = M\frac{a^2}{4}, \quad I_x = M\frac{b^2}{4},$$

and for the principal axis OZ

$$I_z = M\frac{a^2 + b^2}{4}.$$

For a hollow elliptical disk the moment of inertia is equal to that for the whole disk minus that for the hollow space.

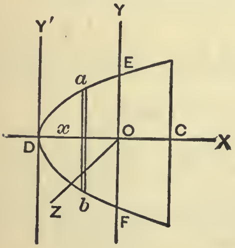
The moment of inertia of a homogeneous ellipse is then the same as for a particle of one eighth the mass of the ellipse at the extremities of the two principal axes and a particle of one half the mass of the ellipse at its centre of mass.

Product of Inertia.—The same holds for product of inertia also. Hence the product of inertia of a homogeneous ellipse with reference to two rectangular axes X, Y, through the centre of mass in the plane of the ellipse is

$$D_{xy} = \frac{M}{4}(a^2 - b^2) \sin\alpha \cos\alpha,$$

and for any two rectangular axes is the same as for a particle of one eighth the mass at the extremities of the two principal axes and a particle of one half the mass at the centre of mass.

[(6) **Moment of Inertia of a Homogeneous Parabola.**—Let c be the chord AB, and h the height CD, and δ the surface density.



Then the mass of the parabola is

$$M = \frac{2}{3}\delta ch,$$

and the distance DO of the centre of mass O from the vertex is

$$DO = \frac{3}{5}h.$$

Let y be the length of an elementary strip ab parallel to the chord c at a distance x from D. Then we have

$$x : y^2 :: h : c^2, \quad \text{or} \quad y = c\sqrt{\frac{x}{h}}.$$

The area of this strip is then $ydx = cdx\sqrt{\frac{x}{h}}$ and its mass is

$$m = \delta cdx\sqrt{\frac{x}{h}}.$$

We have then for the moment of inertia with reference to the line DY'

$$I_{y'} = \int_0^h \delta cx^2dx\sqrt{\frac{x}{h}} = \frac{2\delta ch^3}{7} = \frac{3Mh^2}{7}.$$

For the principal axis OY through the centre of mass we have then

$$I_y = \frac{3Mh^2}{7} - M\left(\frac{3h}{5}\right)^2 = \frac{12}{175}Mh^2.$$

The moment of inertia of the strip ab with reference to the principal axis OX is (page 228) $\frac{my^2}{12}$. Hence

$$I_x = \int_0^h \frac{my^2}{12} = \int_0^h \frac{\delta c^3x^{\frac{1}{2}}dx}{12h\sqrt{h}} = \frac{\delta c^3h}{30} = \frac{Mc^2}{20}.$$

Therefore the moment of inertia with reference to OZ is

$$I_z = I_x + I_y = \frac{1}{5}M\left[\left(\frac{c}{2}\right)^2 + \frac{12}{35}h^2\right].$$

Product of Inertia.—The product of inertia of the elementary strip for two rectangular axes X, Y through its centre of mass, the strip being in the plane XY and making an angle α with the axis of X, is, from page 228,

$$d_{xy} = \frac{my^2}{12}\sin\alpha\cos\alpha = \frac{\delta c^3}{12h^{\frac{3}{2}}}\sin\alpha\cos\alpha\,.\,x^{\frac{3}{2}}dx.$$

The co-ordinates of the centre of mass for two parallel rectangular axes through the centre of mass O are

$$\overline{x} = \left(\frac{3}{5}h - x\right)\cos\alpha, \quad \overline{y} = -\left(\frac{3}{5}h - x\right)\sin\alpha$$

The product of inertia of the strip with reference to these axes is

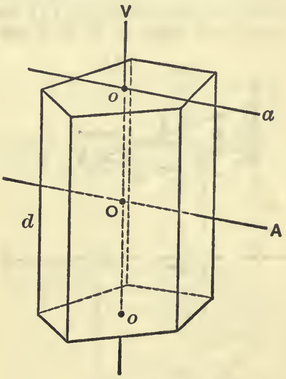
$$d_{xy} + m\,\overline{x}\,\overline{y} = \frac{\delta c^3}{12h^{\frac{3}{2}}}\sin\alpha\cos\alpha\,.\,x^{\frac{3}{2}}dx - \frac{\delta c\sin\alpha\cos\alpha}{h^{\frac{1}{2}}}\left(\frac{3}{5}h - x\right)^2 x^{\frac{1}{2}}dx.$$

Integrating between the limits $x = 0$ and $x = h$, we have for the product of inertia of the parabola with reference to two rectangular axes X, *Y*, through the centre of mass *O* in the plane of the parabola, if the principal axis *DC* makes the angle α with the axis of *X*,

$$D_{xy} = M\left[\frac{12h^2}{175} - \frac{c^2}{20}\right] \sin\alpha\cos\alpha.$$

Equimomental System.—We can easily prove that the moment and product of inertia with reference to any line or pair of rectangular axes is the same as for a system consisting of a particle of $\frac{1}{6}M$ at the extremities *E* and *F* of the axis *OY*, a particle of $\frac{4}{35}M$ at the vertex *D*, a particle of $\frac{6}{35}M$ at the middle of the chord *AB* at *C*, and a particle of $\frac{8}{21}M$ at the centre of mass *O*.

[(7) **Moment of Inertia of a Regular Homogeneous Right Prism in General.**—Let the constant area of cross-section of a right prism be *A*. Take the end planes horizontal and let $d = oo$ be the depth of the prism or length of the axis *oo* through the centres of mass of the end planes.



Take an elementary slice parallel to the end planes, of depth *dy*. The mass of this element, if δ is the density, is

$$m = \delta A dy.$$

Let κ_a be the radius of gyration of the element for any line *oa* in its plane through the centre of mass *o*. Then the moment of inertia of the element with reference to this axis is

$$m\kappa_a^2 = \delta A\kappa_a^2 dy.$$

Take a parallel line *OA* through the centre of mass *O* of the prism at a distance *y* from *oa*. Then the moment of inertia of the element with reference to *OA* is, by the theorem of parallel axes (page 218),

$$M\kappa_a^2 + my^2 = \delta A\kappa_a^2 dy + \delta A y^2 dy.$$

If we integrate between the limits of $y = +\frac{d}{2}$ and $y = -\frac{d}{2}$, we have for the moment of inertia of the prism with reference to any line *OA* through the centre of mass *O* of the prism at right angles to the axis

$$I_a = \delta A\kappa_a^2 d + \delta A\frac{d^3}{12},$$

or, since $\delta A d = M =$ the mass of the prism,

$$I_a = M\kappa_a^2 + M\frac{d^2}{12}. \qquad (1)$$

Let κ_v be the radius of gyration of an elementary slice with reference to a vertical line *oV* through the centre of mass *o*. Then the moment of inertia of every element with reference to this axis is

$$m\kappa_v^2 = \delta A\kappa_v^2 dy.$$

Integrating between the limits $y = +\frac{d}{2}$, $y = -\frac{d}{2}$, we have for the moment of inertia of the prism with reference to the vertical line OV through the centre of mass O of the prism

$$I_v = M\kappa_v^2. \qquad \ldots \ldots \ldots \ldots \qquad (2)$$

These equations are general for any regular homogeneous right prism, whatever the shape of the bases.

(8) **Moment of Inertia of a Regular Homogeneous Right Triangular Prism.**—Let h be the altitude of the triangular bases for any side b. Then we have for the line ob through the centre of mass o of an end parallel to the side b, from page 229,

$$\kappa_b^2 = \frac{h^2}{18}.$$

Hence from equation (1), page 243, for the parallel line OB through the centre of mass O, if M is the mass of the prism,

$$I_b = M\frac{h^2}{18} + M\frac{d^2}{12}. \quad \ldots \quad (1)$$

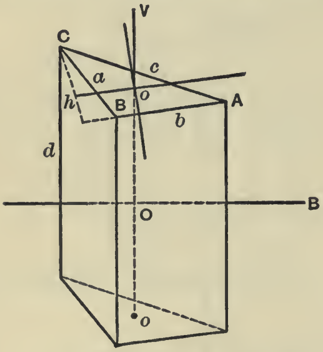
For the vertical line OV through the centre of mass O we have, from page 230,

$$\kappa_v^2 = \frac{1}{36}(a^2 + b^2 + c^2),$$

where a, b, c are the sides respectively opposite A, B, C.

Hence, from equation (2), above,

$$I_v = \frac{M}{36}(a^2 + b^2 + c^2). \quad \ldots \quad (2)$$



For the line through O perpendicular to the plane of BV we have, page 230,

$$\kappa_p^2 = \frac{1}{18}\left(b^2 - \frac{bh}{\tan A} + \frac{h^2}{\tan^2 A}\right).$$

Hence

$$I_p = \frac{M}{18}\left(b^2 - \frac{bh}{\tan A} + \frac{h^2}{\tan^2 A}\right) + M\frac{d^2}{12}. \quad \ldots \quad (3)$$

We see then that the moment of inertia of a regular homogeneous right triangular prism with reference to any line through the centre of mass *is the same as for a particle of* $\frac{1}{18}M$ *at the middle points of each triangular side of each base, and a particle of* $\frac{2}{9}M$ *at the centre of mass of each face.*

COR. 1. If the bases are isosceles triangles we have

$$a^2 = c^2 = h^2 + \frac{b^2}{4}, \quad \text{and} \quad \frac{h}{\tan A} = \frac{b}{2}.$$

Hence

$$I_b = M\frac{h^2}{18} + M\frac{d^2}{12}, \quad I_p = M\frac{b^2}{24} + M\frac{d^2}{12}, \quad I_v = M\frac{b^2}{24} + M\frac{h^2}{18}.$$

Cor. 2. If the bases are equilateral triangles we have

$$a^2 = b^2 = c^2 = \frac{4}{3}h^2, \quad \text{and} \quad \frac{h}{\tan A} = \frac{b}{2} = \frac{c}{2} = \frac{a}{2}.$$

Hence

$$I_b = M\frac{b^2}{24} + M\frac{d^2}{12}, \quad I_p = M\frac{b^2}{24} + M\frac{d^2}{12}, \quad I_v = M\frac{b^2}{12}.$$

Cor. 3. If the bases are right-angled triangles we have

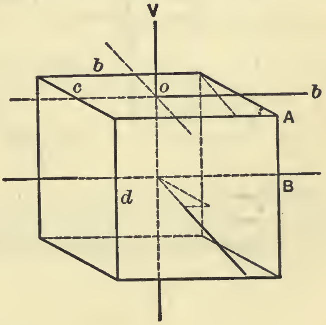
$$a = h, \quad c^2 = h^2 + b^2, \quad \frac{h}{\tan A} = b.$$

Hence

$$I_b = M\frac{h^2}{18} + M\frac{d^2}{12}, \quad I_p = M\frac{b^2}{18} + M\frac{d^2}{12}, \quad I_v = \frac{M}{18}(h^2 + b^2) = M\frac{c^2}{18}.$$

(9) Moment of Inertia of a Homogeneous Right Parallelopipedon.—Let h be the altitude of the base for any side b. Then we have for the line ob through the centre of mass o of an end parallel to the side b, from page 238,

$$\kappa_b^2 = \frac{h^2}{12}.$$



Hence from equation (1), page 243, for the parallel line OB through the centre of mass O, if M is the mass of the prism,

$$I_b = \frac{M}{12}(h^2 + d^2). \quad . \quad . \quad (1)$$

For the vertical line OV through the centre of mass O we have, from page 238,

$$\kappa_v^2 = \frac{1}{12}(b^2 + c^2).$$

Hence from equation (2), page 244,

$$I_v = \frac{M}{12}(b^2 + c^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the line through O perpendicular to the plane of BV we have, page 236,

$$\kappa_p^2 = \frac{1}{12}(b^2 + c^2 \cos^2 A).$$

Hence

$$I_p = \frac{M}{12}(b^2 + d^2 + c^2 \cos^2 A). \quad . \quad . \quad . \quad . \quad (3)$$

We see then that the moment of inertia of a homogeneous right parallelopipedon *is the same as for a particle of* $\frac{1}{6}M$ *at the centre of mass of each face.*

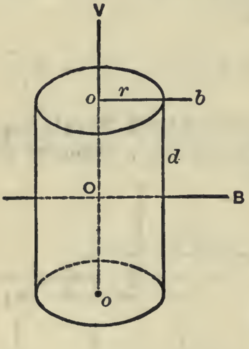
COR. If the bases are rectangular, $A = 90°$, $c = h$, and we have

$$I_b = \frac{M}{12}(h^2 + d^2), \quad I_p = \frac{M}{12}(b^2 + d^2), \quad I_v = \frac{M}{12}(b^2 + h^2).$$

For a cube, $h = b = d$, and

$$I_b = \frac{Md^2}{6}, \quad I_p = \frac{Md^2}{6}, \quad I_v = \frac{Md^2}{6}.$$

(10) Moment of Inertia of a Homogeneous Right Cylinder.—Let r be the radius of the circular base and d the depth or length.



Then for any line ob through the centre of mass o of an end we have, from page 240,

$$\kappa_b^2 = \frac{r^2}{4}.$$

Hence from equation (1), page 243, for a parallel line through the centre of mass O, if M is the mass of the cylinder,

$$I_b = M\frac{r^2}{4} + M\frac{d^2}{12}. \quad . \quad . \quad . \quad . \quad (1)$$

For the geometrical axis OV we have, page 240,

$$\kappa_v^2 = \frac{r^2}{2}.$$

Hence from equation (2), page 244,

$$I_v = M\frac{r^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For a hollow circular cylinder, if r_1 is the outside and r_2 the inside radius, we have, page 240, $\kappa_b^2 = \frac{1}{4}(r_1^2 + r_2^2)$ and $\kappa_v^2 = \frac{1}{2}(r_1^2 + r_2^2)$. Hence

$$I_b = \frac{M}{4}(r_1^2 + r_2^2) + M\frac{d^2}{12}, \quad I_v = \frac{M}{2}(r_1^2 + r_2^2). \quad . \quad . \quad . \quad (3)$$

If the bases are ellipses, let the semi-transverse axis be a and the semi-conjugate axis b. Then (page 241) we have $\kappa_b^2 = \frac{a^2}{4}$, $\kappa_a^2 = \frac{b^2}{4}$, $\kappa_v^2 = \frac{a^2+b^2}{4}$. Hence

$$I_a = M\frac{b^2}{4} + M\frac{d^2}{12}, \quad I_b = M\frac{a^2}{4} + M\frac{d^2}{12}, \quad I_v = M\frac{a^2 + b^2}{4}. \quad . \quad . \quad (4)$$

For a hollow elliptical cylinder the moment of inertia is equal to that for the whole cylinder minus that for the hollow space.

If the bases are parabolas, let c be the chord AB, and h the height CD of the parabolic bases (page 241). Then we have for the lines at right angles to the axis through the centre of mass parallel to h and c, and for the vertical line through the centre of mass,

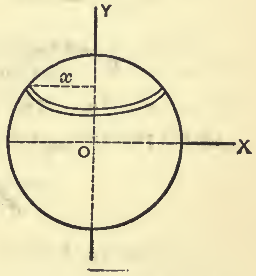
$$\kappa_h^2 = \frac{c^2}{20}, \quad \kappa_c^2 = \frac{12}{175}h^2, \quad \kappa_v^2 = \frac{c^2}{20} + \frac{12}{175}h^2. \quad . \quad . \quad . \quad (5)$$

Hence

$$I_h = M\frac{c^2}{20} + M\frac{d^2}{12}, \quad I_c = \frac{12}{175}Mh^2 + M\frac{d^2}{12}, \quad I_v = M\left(\frac{c^2}{20} + \frac{12h^2}{175}\right). \quad . \quad (6)$$

[(11) **Moment of Inertia of a Homogeneous Sphere.**—Let r be the radius, and take a circular slice at a distance y from OX. The radius x of this slice is given by

$$x^2 = r^2 - y^2.$$



The area of the slice is then $\pi x^2 = \pi(r^2 - y^2)$. Its volume is $\pi dy(r^2 - y^2)$, and its mass, if δ the density, is

$$m = \delta\pi dy(r^2 - y^2).$$

The moment of inertia of the slice about OY is then, page 240,

$$\frac{mx^2}{2} = \frac{\delta\pi}{2}(r^2 - y^2)^2 dy. \quad . \quad . \quad . \quad (1)$$

Integrating (1) between the limits $y = +r$ and $y = 0$, we have for the moment of inertia *of a hemisphere* with reference to the line OY perpendicular to its base at the centre, since the mass of the hemisphere is $M = \frac{2}{3}\delta\pi r^3$,

$$I_y = \frac{2}{5}Mr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The moment of inertia of the slice about OX is

$$\frac{mx^2}{4} + my^2 = \frac{\delta\pi}{4}(r^2 - y^2)^2 dy + \delta\pi(r^2 - y^2)y^2 dy. \quad . \quad . \quad . \quad (3)$$

Integrating (3) between the limits $y = +r$ and $y = 0$, we have for the moment of inertia *of a hemisphere* with reference to any line OX in its base through the centre

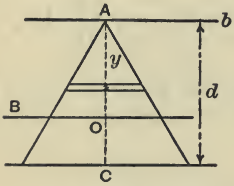
$$I_x = \frac{2}{5}Mr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Integrating (1) or (3) between the limits $y = +r$, $y = -r$, we have, since $M = \frac{4}{3}\delta\pi r^3$ is the mass of the sphere, for the moment of inertia of the entire sphere with reference to any line through the centre

$$I_y = I_x = \frac{2}{5}Mr^2.$$

COR. The formula $I = \frac{2}{5}Mr^2$ evidently holds good also for any *spheroid* of revolution whose equatorial radius is r.

(12) Moment of Inertia of a Homogeneous Right Cone or Pyramid.—Let A be the area of the base, d the depth or altitude. Take any slice parallel to the base at a distance y from the vertex. Then the area of this slice is $\frac{y^2}{d^2}A$, its volume is $\frac{Ay^2dy}{d^2}$, and if δ is the density its mass is



$$m = \frac{\delta Ay^2dy}{d^2}.$$

Let κ_b be the radius of gyration of the base for any line in the plane of the base. Then the radius of gyration of the slice for any parallel line in its plane is $\frac{y}{d}\kappa_b$. The moment of inertia of the slice with reference to a parallel line through the vertex A is then

$$\frac{my^2\kappa_b^2}{d^2} + my^2 = \frac{\delta A\kappa_b^2y^4dy}{d^4} + \frac{\delta Ay^4dy}{d^2}.$$

Integrating between the limits $y = d$ and $y = 0$, we have, since $\frac{\delta Ad}{3} = M$ is the mass of the cone or pyramid, for the moment of inertia for a line Ab at right angles to the axis

$$I_b' = \frac{3}{5}M(\kappa_b^2 + d^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For a parallel axis through the centre of mass O

$$I_b = I_b' - M\left(\frac{3}{4}d\right)^2 = \frac{3}{5}M\kappa_b^2 + \frac{3}{80}Md^2. \quad . \quad . \quad . \quad . \quad (2)$$

Let κ_v be the radius of gyration of the base for the axis AC. Then the radius of gyration of the slice for this axis is $\frac{y}{d}\kappa_v$. The moment of inertia of any slice with reference to the axis AC is then

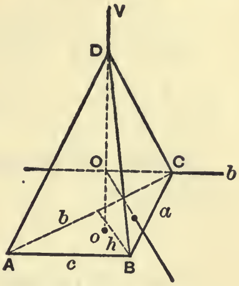
$$\frac{my^2\kappa_v^2}{d^2} = \frac{\delta A\kappa_v^2y^4dy}{d^4}.$$

Integrating between the limits $y = d$ and $y = 0$, we have for the moment of inertia for the axis AC

$$I_v = \frac{3}{5}M\kappa_v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These equations hold good for any base; we have only to substitute for κ_b and κ_v their values in any case.

1. Right Pyramid with Triangular Base.—Thus for a right pyramid with triangular base we have (page 229) for a line through the centre of mass *o* of the base parallel to *b*



$$\kappa_b^2 = \frac{h^2}{18},$$

where *h* is the altitude of the triangular base for the side *b*. Hence for a parallel line *Ob* through the centre of mass *O* we have, from (2),

$$I_b = M\frac{h^2}{30} + \frac{3}{80}Md^2. \quad . \quad . \quad . \quad (4)$$

For a line through *o* at right angles to *b* we have (page 230)

$$\kappa_p^2 = \frac{1}{18}\left(b^2 - \frac{bh}{\tan A} + \frac{h^2}{\tan^2 A}\right).$$

Hence for a parallel line through *O*, from (2),

$$I_p = \frac{M}{30}\left(b^2 - \frac{bh}{\tan A} + \frac{h^2}{\tan^2 A}\right) + \frac{3}{80}Md^2. \quad . \quad . \quad . \quad . \quad (5)$$

For the axis *Do* we have (page 230)

$$\kappa_v^2 = \frac{1}{36}(a^2 + b^2 + c^2).$$

Hence for the axis *Do*

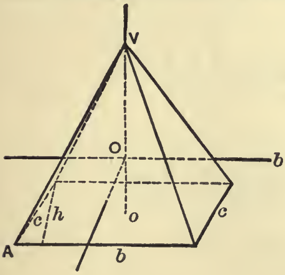
$$I_v = \frac{M}{60}(a^2 + b^2 + c^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For an isosceles triangle for base, $a^2 = c_1^2 = h^2 + \frac{b^2}{4}$ and $\frac{h}{\tan A} = \frac{b}{2}$.

For an equilateral triangle for base, $a^2 = b^2 = c^2 = \frac{4}{3}h^2$ and $\frac{h}{\tan A} = \frac{b}{2} = \frac{c}{2} = \frac{a}{2}$.

For a right-angled triangle for base, $a = h$, $c^2 = h^2 + b^2$, $\frac{h}{\tan A} = b$.

2. Right Pyramid with Parallelogram Base.—For a right pyramid with parallelogram base we have (page 229) for a line through the centre of mass *o* of the base parallel to *b*



$$\kappa_b^2 = \frac{h^2}{12},$$

where *h* is the altitude of the base for the side *b*. Hence for a parallel line *Ob* through the centre of mass *O* we have, from (2),

$$I_b = M\frac{h^2}{20} + \frac{3}{80}Md^2. \quad . \quad . \quad (7)$$

For a line through o at right angles to b we have (page 236)

$$\kappa_p^2 = \frac{1}{12}(b^2 + c^2 \cos^2 A).$$

Hence for a parallel line through O

$$I_p = \frac{M}{20}(b^2 + c^2 \cos^2 A) + \frac{3}{80}Md^2. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For the axis Do we have (page 236)

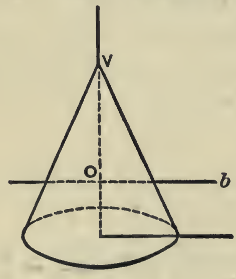
$$\kappa_v^2 = \frac{1}{12}(b^2 + c^2).$$

Hence for the axis

$$I_v = \frac{M}{20}(b^2 + c^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For a rectangular base $A = 90°$ and $\cos A = 0$.
For a square base $c = b$ and $b = c = h$.

3. Right Cone with Circular Base.—For a right cone with circular base we have (page 239) for any line in the plane of the base through its centre of mass



$$\kappa_b^2 = \frac{r^2}{4}.$$

Hence for a parallel line Ob through the centre of mass O

$$I_b = \frac{3}{20}Mr^2 + \frac{3}{80}Md^2. \quad . \quad . \quad . \quad (10)$$

For the axis OV we have

$$\kappa_v^2 = \frac{r^2}{2}.$$

Hence for the axis OV

$$I_v = \frac{3}{10}Mr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

4. Right Cone with Elliptic Base.—If the semi-axes of the base are a and b, we have (page 241)

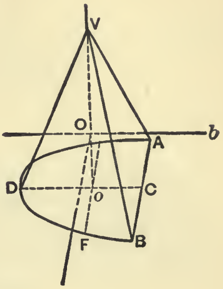
$$\kappa_a^2 = \frac{b^2}{4}, \qquad \kappa_b^2 = \frac{a^2}{4}, \qquad \kappa_v^2 = \frac{a^2 + b^2}{4}.$$

Hence

$$I_a = \frac{3}{20}b^2M + \frac{3}{80}Md^2, \qquad I_b = \frac{3}{20}Ma^2 + \frac{3}{80}Md^2, \quad . \quad . \quad . \quad (12)$$

$$I_v = \frac{3}{20}(Ma^2 + b^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

5. Right Cone with Parabolic Base.—If c is the chord AB, and h the height CD of the base, then for the line oC we have (page 241)



$$\kappa_b^2 = \frac{c^2}{20}.$$

For the line oF

$$\kappa_p^2 = \frac{12}{175}h^2.$$

For the axis OV

$$\kappa_v^2 = \frac{c^2}{20} + \frac{12h^2}{175}.$$

Hence

$$I_b = \frac{3}{100}Mc^2 + \frac{3}{80}Md^2, \qquad I_p = \frac{36}{875}Mh^2 + \frac{3}{80}d^2M, \quad . \quad . \quad (14)$$

$$I_v = \frac{3}{100}Mc^2 + \frac{36}{875}Mh^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

CHAPTER III.

MOTION IN TWO DIMENSIONS.

MOTION OF A BODY IN GENERAL. MOTION IN A PLANE. GENERAL FORMULAS FOR MOTION IN TWO DIMENSIONS.

Motion of a Body in General.—We have already discussed (Chap. I, page 167) the rotation of a body about a fixed axis.

Now we have seen (Vol. II, *Statics*, page 83) *that the motion of the centre of mass of a body is the same as if all the mass and all the forces were collected at the centre of mass.*

The motion of a body in general consists then of rotation about an axis through the centre of mass and of translation of the centre of mass.

The rotation about the centre of mass is the same as if the centre of mass were fixed. So far, then, as motion of translation is concerned we can treat the body as a particle at the centre of mass and consider all forces acting upon the body as acting upon this particle without change in magnitude or direction.

So far as motion of rotation is concerned we may consider the centre of mass as fixed.

Motion in a Plane.—Let all the forces acting upon a body be coplanar and the initial velocity of the centre of mass be in the same plane. Then the centre of mass will move always in this plane and can be treated as a particle, and the rotation about the centre of mass will take place in the same plane about an axis perpendicular to the plane through the centre of mass, the same as if the axis were fixed.

We have then motion in two dimensions only.

If M is the mass of the body, $\bar{f}$ the acceleration of the centre of mass in any direction, and F the resultant force in that direction, we have

$$M\bar{f} = F = \text{force}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If $\bar{v}$ is the velocity of the centre of mass in any direction, we have for the momentum in that direction

$$M\bar{v} = \text{momentum}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If ϕ is the impulse and v_1, v the initial and final velocities of the centre of mass in any direction, we have

$$M(\bar{v} - \bar{v}_1) = \phi = \text{impulse.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The kinetic energy of translation in any direction is

$$E = \frac{1}{2}M\bar{v}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Let F be any force, p its lever-arm with reference to the axis through the centre of mass at right angles to the plane of motion, and I the moment of inertia of the body with reference to this axis. Then if α is the angular acceleration about this axis, we have (page 170), since, so far as rotation is concerned, we can consider the centre of mass as fixed,

$$I\alpha = \Sigma Fp = \textit{moment} \text{ of forces causing rotation.} \quad . \quad . \quad (I)$$

Let m be the mass of a particle and v its velocity and r the lever-arm of the velocity with reference to the axis through the centre of mass at right angles to the velocity. Then we have (page 171)

$$I\omega = \Sigma mvr = \textit{moment} \text{ of momentum of rotation.} \quad . \quad . \quad (II)$$

Also if ω_1 is the initial and ω the final angular velocity,

$$I(\omega - \omega_1) = \Sigma \phi r = \textit{moment} \text{ of the impulse of rotation.} \quad . \quad . \quad (III)$$

The kinetic energy of rotation (page 171) is

$$E = \frac{1}{2}I\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IV)$$

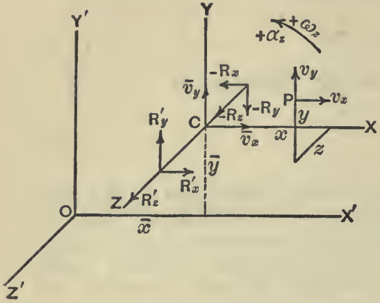
We again call the attention of the student to the analogy between equations (1), (2), (3) and (4) for rectilinear motion, and the corresponding equations (I), (II), (III) and (IV) for rotation, and to the part played by the quantity $I = \Sigma mr^2$, which we have called the moment of inertia.

We see that in the equations (1), (2) and (3) for force, momentum and impulse for rectilinear motion, if we replace mass M by moment of inertia I, and linear acceleration and velocity f and v by angular acceleration and velocity α and ω, we obtain *moment* of force, momentum and impulse for rotation.

Also, if in the equation for kinetic energy for rectilinear motion we replace M by moment of inertia I and linear velocity v by angular velocity ω, we obtain kinetic energy of rotation.

General Formulas for Motion in Two Dimensions.—Let a rigid body of mass M rotate about an axis CZ through the centre of mass C with the angular velocity ω_z and the angular acceleration $\alpha_z = \frac{d\omega_z}{dt}$, positive direction of rotation being counter-clockwise as in the figure.

Pass a plane through C at right angles to the axis CZ, and take the other two co-ordinate axes X, Y in this plane. Then the plane of XY is the plane of rotation. Let the body be so constrained that the centre of mass C moves always in this plane and the axis CZ is always at right angles to it. We have then motion in two dimensions only, and the centre of mass C is a moving origin fixed in the body and moving with it.



Take a fixed origin O in space, in the plane of XY, and take the co-ordinate axes OX', OY', OZ' parallel to CX, CY, CZ.

Let the co-ordinates of C with reference to O be $\bar{x}$, $\bar{y}$, and the co-ordinates of any point P of the body in general with reference to O be x', y', z', and with reference to C, x, y, z.

In the same way let the components of the velocity of C with reference to O be $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z = 0$ and the components of the velocity of any point P of the body in general with reference to O be v_x', v_y', $v_z' = 0$, and with reference to C, v_x, v_y, $v_z = 0$.

So also let the components of the acceleration of C with reference to O be $\bar{f}_x$, $\bar{f}_y$, $\bar{f}_z = 0$, and the components of the acceleration of any point P of the body in general with reference to O be f_x', f_y', $f_z' = 0$, and with reference to C, f_x, f_y, $f_z = 0$.

Then we have by reason of our notation

$$x' = \bar{x} + x, \quad y' = \bar{y} + y, \quad z' = z.$$

For the components v_x, v_y, v_z of P due to rotation about CZ we have

$$\frac{dx}{dt} = v_x = -y\omega_z, \quad \frac{dy}{dt} = v_y = x\omega_z, \quad \frac{dz}{dt} = v_z = 0. \quad . \quad (1a)$$

At the centre of mass C there is no velocity due to rotation, but only velocity of translation. Every point of the body not in the axis CZ has this velocity of translation and also a velocity due to rotation about CZ. We have then for the combined velocity of any point P of the body with reference to O

$$\left.\begin{aligned} \frac{dx'}{dt} &= v_x' = \bar{v}_x + v_x = \bar{v}_x - y\omega_z; \\ \frac{dy'}{dt} &= v_y' = \bar{v}_y + v_y = \bar{v}_y + x\omega_z; \\ \frac{dz'}{dt} &= v_z' = \bar{v}_z + v_z = 0. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

If in (1) we make $\bar{v}_x = 0$, $\bar{v}_y = 0$, $y = y'$, $x = x'$, we have equations (1), page 190, for rotation only about a fixed axis OZ'.

We have for the components of the tangential acceleration of P with reference to C, due to rotation only about CZ,

$$f_{tx} = -y\alpha_z, \quad f_{ty} = +x\alpha_z, \quad f_{tz} = 0,$$

and for the components of the normal acceleration of P with reference to C, due to rotation only about CZ,

$$f_{nx} = -x\omega_z^2, \quad f_{ny} = -y\omega_z^2, \quad f_{nz} = 0.$$

At the centre of mass there is no acceleration due to rotation, but only acceleration of translation. Every point of the body not in the axis CZ

has this acceleration of translation and also the accelerations due to rotation. We have then for the combined acceleration with reference to O of any point P

$$\left.\begin{aligned} \frac{d^2x'}{dt^2} &= f_x' = \bar{f}_x + f_{tx} + f_{nx} = \bar{f}_x - y\alpha_z - x\omega_z^2; \\ \frac{d^2y'}{dt^2} &= f_y' = \bar{f}_y + f_{ty} + f_{ny} = \bar{f}_y + x\alpha_z - y\omega_z^2; \\ \frac{d^2z'}{dt^2} &= f_z' = \bar{f}_z + f_{tz} + f_{nz} = 0. \end{aligned}\right\} \quad . \; . \; . \quad (2)$$

If in (2) we make $\bar{f}_x = 0$, $\bar{f}_y = 0$, and $y = y'$, $x = x'$, we have equations (2), page 191, for rotation only about a fixed axis OZ'.

We can evidently obtain (2) directly from (1) by differentiating, since

$$\frac{d\omega_z}{dt} = \alpha_z, \quad \frac{d\bar{v}_x}{dt} = \bar{f}_x, \quad \frac{dy}{dt} = x\omega_z = v_y, \text{ etc.}$$

Since CZ passes through the centre of mass, we have, if m is the mass of a particle,

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0. \quad . \; . \; . \; . \; . \quad (3)$$

Also

$$\Sigma m = M, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

and

$$\left.\begin{aligned} \Sigma m(x^2 + y^2) &= I_z = \text{moment of inertia for } CZ; \\ \Sigma m(x'^2 + y'^2) &= I_z' = \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“ } OZ'. \end{aligned}\right\} \quad . \; . \quad (5)$$

Motion of Centre of Mass.—From (2) we have for the sum of the components of all the effective forces, after reduction by (3) and (4),

$$\Sigma mf_x' = M\bar{f}_x, \quad \Sigma mf_y' = M\bar{f}_y, \quad \Sigma mf_z' = 0. \; . \; . \; . \; . \quad (6)$$

But by D'Alembert's principle (page 168) the sum of the components of the impressed forces in any direction is equal to the sum of the components of the effective forces in that direction.

Hence, *the centre of mass moves at any instant as if all the mass and impressed forces were collected at the centre of mass.*

Momentum.—From (1) we have for the sum of the components of momentum of all the particles, after reduction by (3) and (4),

$$\Sigma mv_x' = M\bar{v}_x, \quad \Sigma mv_y' = M\bar{v}_y, \quad \Sigma mv_z' = 0. \quad . \; . \; . \quad (7)$$

Hence, *the momentum of the body is the same as for all the mass collected at the centre of mass.*

Moment of Momentum.—Let $\mathfrak{M}'_{mx}$, $\mathfrak{M}'_{my}$, $\mathfrak{M}'_{mz}$ be the sums of the moments of momentum of all the particles about the co-ordinate axes OX', OY', OZ', and $\mathfrak{M}_{mx}$, $\mathfrak{M}_{my}$, $\mathfrak{M}_{mz}$ for the co-ordinate axes CX, CY, CZ for any axis of rotation CZ fixed in direction so as always to be at right angles to the plane of motion $X'Y'$. Then we have from (1), after reduction by (3), (4), and (5),

$$\left.\begin{aligned} \mathfrak{M}'_{mx} &= \Sigma m(v_z'y' - v_y'z') = -\,\omega_z\Sigma mxz = \mathfrak{M}_{mx}; \\ \mathfrak{M}'_{my} &= \Sigma m(v_x'z' - v_z'x') = -\,\omega_z\Sigma myz = \mathfrak{M}_{my}; \\ \mathfrak{M}'_{mz} &= \Sigma m(v_y'x' - v_x'y') = I_z'\omega_z + M\bar{v}_y\bar{x} - M\bar{v}_x\bar{y}. \end{aligned}\right\} \quad . \; . \quad (8)$$

In the last of these equations the first term on the right is the moment of momentum $\mathfrak{M}_{mz}$ for rotation about CZ, and the other two terms on the right give the moment of momentum about OZ' due to translation. If there is no translation, so that the body rotates only about the fixed axis CZ, we have $\overline{v_x} = 0$, $\overline{v_y} = 0$, and hence

$$\mathfrak{M}_{mz} = I_z \omega_z.$$

This is equation (II), page 172. If the fixed axis of rotation is OZ' we have, from (1a), $\overline{v_y} = \bar{x}\omega_z$, $\overline{v_x} = -\bar{y}\,\omega_z$, and hence, since $I_z + M(\bar{x}^2 + \bar{y}^2) = I_z'$,

$$\mathfrak{M}'_{mz} = I_z' \omega_z,$$

as on page 192, for rotation about a fixed axis.

The resultant moment of momentum is in general given by

$$\mathfrak{M}'_m = \sqrt{\mathfrak{M}'^2_{mx} + \mathfrak{M}'^2_{my} + \mathfrak{M}'^2_{mz}}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the direction-cosines of the resultant axis of moment of momentum are

$$\frac{\mathfrak{M}'_{mx}}{\mathfrak{M}'_m}, \quad \frac{\mathfrak{M}_{my}}{\mathfrak{M}'_m}, \quad \frac{\mathfrak{M}'_{mz}}{\mathfrak{M}'_m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the axis CZ is a principal axis we have $\Sigma mxz = 0$, $\Sigma myz = 0$, and hence $\mathfrak{M}'_{mx} = 0$, $\mathfrak{M}'_{my} = 0$. Hence if the axis CZ is a principal axis the resultant axis of momentum coincides in direction with CZ, otherwise it makes an angle with CZ.

Kinetic Energy.—Let v' be the velocity of any particle and $\bar{v}$ the velocity of the centre of mass C with reference to O, so that

$$v'^2 = v_x'^2 + v_y'^2, \qquad \bar{v}^2 = \overline{v_x}^2 + \overline{v_y}^2.$$

Then we have, from (1),

$$\frac{1}{2} m v_x'^2 = \frac{1}{2} m \overline{v_x}^2 - m \overline{v_x}\, y \omega_z + \frac{1}{2} m y^2 \omega_z^2;$$

$$\frac{1}{2} m v_y'^2 = \frac{1}{2} m \overline{v_y}^2 + m \overline{v_y}\, x \omega_z + \frac{1}{2} m x^2 \omega_z^2.$$

Adding these, we have for the sum of the kinetic energy of all the particles

$$E' = \Sigma \frac{1}{2} m v'^2 = \frac{1}{2} \bar{v}^2 \Sigma m + \overline{v_y} \omega_z \Sigma mx - \overline{v_x} \omega_z \Sigma my + \frac{1}{2} \omega_z^2 \Sigma m(x^2 + y^2),$$

or, reducing by (3), (4) and (5),

$$E' = \frac{1}{2} M \bar{v}^2 + \frac{1}{2} I_z \omega_z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The first term on the right is the kinetic energy of translation for the total mass collected at the centre of mass, and the second on the right is the kinetic energy for rotation about CZ. If there is no translation, so that the body rotates only about the fixed axis CZ, we have

$$E = \frac{1}{2} I_z \omega_z^2.$$

This is equation (IV), page 172. If the fixed axis of rotation is OZ' we have

$$E' = \frac{1}{2} I_z' \omega_z^2,$$

as on page 193 for rotation about a fixed axis.

Moment of the Effective Forces.—Let $\mathfrak{M}'_{fx}$, $\mathfrak{M}'_{fy}$, $\mathfrak{M}'_{fz}$ be the sums of the moments of the effective forces about the co-ordinate axes OX', OY', OZ'. Then we have from (2), after reduction by (3), (4) and (5),

$$\left.\begin{aligned}\mathfrak{M}'_{fx} &= \Sigma m(f_z'y' - f_y'z') = -\alpha_z \Sigma mxz + \omega_z^2 \Sigma myz;\\ \mathfrak{M}'_{fy} &= \Sigma m(f_x'z' - f_z'x') = -\alpha_z \Sigma myz - \omega_z^2 \Sigma mxz;\\ \mathfrak{M}'_{fz} &= \Sigma m(f_y'x' - f_x'y') = I_z\alpha_z + M\bar{f}_y\bar{x} - M\bar{f}_x\bar{y}.\end{aligned}\right\} \quad . \quad (12)$$

In the last of these equations the first term on the right is the moment of the effective forces for rotation about CZ, and the other two terms give the moment for translation of the effective force of the entire mass at the centre of mass.

In the first two equations (12) we have (page 193)

$$\Sigma m(f_{tz}y - f_{ty}z) = -\alpha_z \Sigma mxz, \quad \Sigma m(f_{tx}z - f_{tz}x) = -\alpha_z \Sigma myz.$$

These terms therefore give the moments about CX, CY of the effective forces due to the tangential accelerations or the effective tangential forces of the particles. We have also

$$\Sigma m(f_{nz}y - f_{ny}z) = +\omega_z^2 \Sigma myz, \quad \Sigma m(f_{nx}z - f_{nz}x) = -\omega_z^2 \Sigma mxz.$$

These terms therefore give the moments about CX, CY of the effective forces due to the normal accelerations, or the effective deflecting forces of the particles.

If there is no translation, so that the body rotates only about a fixed axis CZ, we have

$$\mathfrak{M}_{fz} = I_z\alpha_z.$$

This is equation (I), page 172. If the fixed axis of rotation is OZ', we have

$$\mathfrak{M}'_{fz} = I_z'\alpha_z,$$

as on page 193, for rotation about a fixed axis.

If the axis CZ is a principal axis we have $\mathfrak{M}'_{fx} = 0$, $\mathfrak{M}'_{fy} = 0$, or the moment of the effective tangential and deflecting forces about OX' and OY' is zero.

External Forces.—Conceive the body to be fixed to the axis CZ at two points distant a' and a'' from C, and let the reaction of these points on the body resolved parallel to the co-ordinate axes be respectively $-R_x$, $-R_y$, $+R_z''$ and $+R_x$, $+R_y$, $+R_z'$. (See figure, page 254.)

These forces are all impressed forces; but since they are internal to the system consisting of the body and other bodies, we call them *internal* forces.

All *other* impressed forces acting upon the body we call therefore *external* forces.

Let these external forces be F_1, F_2, F_3, etc., making the angles $(\alpha_1$, β_1,

γ_1), (α_2, β_2, γ_2), etc., with the co-ordinate axes. Then we have for the resultant components of the external forces

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \text{etc.} = \Sigma F \cos \alpha; \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \text{etc.} = \Sigma F \cos \beta; \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \text{etc.} = \Sigma F \cos \gamma. \end{aligned}\right\} \quad . \quad . \quad (13)$$

Moment of the External Forces. — Let $\mathfrak{M}_{ex}$, $\mathfrak{M}_{ey}$, $\mathfrak{M}_{ez}$ be the sums of the moments of the external forces about the co-ordinate axes CX, CY, CZ, and let (x_1, y_1, z_1), (x_2, y_2, z_2), etc., be the co-ordinates of the points of application of the external forces F_1, F_2, etc. Then we have

$$\left.\begin{aligned} \mathfrak{M}_{ex} &= \Sigma Fy \cos \gamma - \Sigma Fz \cos \beta; \\ \mathfrak{M}_{ey} &= \Sigma Fz \cos \alpha - \Sigma Fx \cos \gamma; \\ \mathfrak{M}_{ez} &= \Sigma Fx \cos \beta - \Sigma Fy \cos \alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Pressures on the Axis.—We have by D'Alembert's principle (page 168) the resultant of the impressed equal to the resultant of the effective forces, or, from (6),

$$\left.\begin{aligned} F_x &= \Sigma m f_x' = M\bar{f}_x; \\ F_y &= \Sigma m f_y' = M\bar{f}_y; \\ F_z + R_z' + R_z'' &= \Sigma m f_z' = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Also taking moments about the co-ordinate axes CX, CY, CZ, we have by D'Alembert's principle, from (12) and (14),

$$\left.\begin{aligned} &\Sigma Fy \cos \gamma - \Sigma Fz \cos \beta - R_y(a' + a'') \\ &\qquad = \mathfrak{M}_{fx} = - \alpha_z \Sigma mxz + \omega_z^2 \Sigma myz; \\ &\Sigma Fz \cos \alpha - \Sigma Fx \cos \gamma + R_x(a' + a'') \\ &\qquad = \mathfrak{M}_{fy} = - \alpha_z \Sigma myz - \omega_z^2 \Sigma mxz; \\ &\Sigma Fx \cos \beta - \Sigma Fy \cos \alpha \qquad = \mathfrak{M}_{fz} = I_z \alpha_z. \end{aligned}\right\} \quad (16)$$

From the last of these equations we can find α_z, and then from the first two we can find R_y and R_x. Then R_z' and R_z'' are indeterminate, but their sum is given by the last of equations (15), viz., $R_z = R_z' + R_z''$.

If there are no external forces, or if all the *impressed* forces pass through the centre of mass, we have in either case, from the last of equations (16), $\alpha_z = 0$ and all terms containing α_z disappear. Now $+ \omega_z^2 \Sigma myz$ and $- \omega_z^2 \Sigma mxz$ (page 195) are the moments about X and Y of the deflecting forces of the particles. If CZ is a principal axis, we have, taking the other two principal axes as the co-ordinate axes, $\Sigma myz = 0$, $\Sigma mxz = 0$, and the moments of the deflecting forces are zero.

Hence, *if a body rotates about a fixed principal axis through the centre of mass there is no stress on that axis due to the deflecting forces.*

Spontaneous Axis of Rotation.—The axis through the centre of mass about which a body rotates at any instant we call the **spontaneous axis** of rotation.* In the case of motion in two dimensions it is of course always at right angles to the plane of rotation.

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* Usually called the "instantaneous axis." We prefer the term *spontaneous axis.*

Instantaneous Axis.—The axis *fixed in space* about which a body rotates at any instant when the centre of mass *moves in a plane at right angles to the spontaneous axis* we call the **instantaneous axis** of rotation.*

If in (1) we make $v_x' = 0$, $v_y' = 0$, we have for the co-ordinates x'', y'' *with reference to C* of a point whose velocity is zero with reference to O

$$\overline{v_x} - y''\omega_z = 0,$$

$$\overline{v_y} + x''\omega_z = 0.$$

Evidently every point of a straight line through this point parallel to CZ is at rest at the instant. This line is the instantaneous axis. Its co-ordinates with reference to C are then

$$y'' = \frac{\overline{v_x}}{\omega_z}, \qquad x'' = -\frac{\overline{v_y}}{\omega_z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

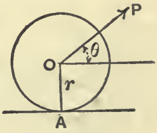
In the same way if $\overline{f_{tx}}, \overline{f_{ty}}$ are the components of the tangential acceleration of C due to rotation about the instantaneous axis, we have for the co-ordinates of this axis with reference to C

$$y'' = \frac{\overline{f_{tx}}}{\alpha_z}, \qquad x'' = -\frac{\overline{f_{ty}}}{\alpha_z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

EXAMPLES.

(1) *A rigid homogeneous circular disk of mass M whose plane is vertical has a force of P lbs. applied at the centre and rolls without sliding upon a rigid horizontal plane. Determine its motion.*

Ans. Let the force P make the angle θ with the horizontal. If P is the force in pounds, the force in poundals is Pg. The horizontal component is $Pg \cos \theta$ and the vertical component is $Pg \sin \theta$.



Let r be the radius and A the point of contact. The moment of the impressed forces about A is $-Pg \cos \theta \,.\, r$. Let κ be the radius of gyration for the axis through the centre of mass C at right angles to the plane of the disk, and I' the moment of inertia with reference to a parallel line through A. Then $I' = M(\kappa^2 + r^2)$, and we have (page 253), if α is the angular acceleration,

$$I'\alpha = -Pgr \cos \theta, \quad \text{or} \quad \alpha = -\frac{Pgr \cos \theta}{M(\kappa^2 + r^2)}.$$

The axis at A is the instantaneous axis, hence (equation (18), page 259) the linear acceleration of the centre is

$$\overline{f_{tx}} = -r\alpha = \frac{Pgr^2 \cos \theta}{M(\kappa^2 + r^2)}.$$

For a disk $\kappa^2 = \frac{r^2}{2}$ (page 239), hence

$$\alpha = -\frac{2Pg \cos \theta}{3Mr}, \overline{f_{tx}} = \frac{2Pg \cos \theta}{3M}.$$

Both angular and linear tangential accelerations are then constant if P is constant, and the displacement and velocity of the centre of mass and the angular velocity of the disk after any given time may be readily determined.

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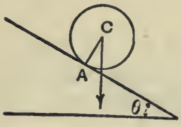
* Usually called "spontaneous axis." We prefer the term *instantaneous axis.*

(2) *A rigid homogeneous circular disk of mass M whose plane is vertical rolls (without sliding) down a rigid inclined plane. Determine its motion.*

Ans. Let the radius be r, the radius of gyration about an axis through the centre of mass C perpendicular to the plane of the disk be κ, and θ the inclination of the plane, and A the point of contact.



Then the weight is Mg, the force parallel to the plane is $Mg \sin \theta$, and its moment about A, $- Mgr \sin \theta$. We have then, as in the preceding example,

$$I'\alpha = M(\kappa^2 + r^2)\alpha = - Mgr \sin \theta, \quad \text{or} \quad \alpha = -\frac{gr \sin \theta}{\kappa^2 + r^2}.$$

Also, since A is the instantaneous axis, the linear acceleration of the centre is

$$\bar{f}_{tx} = - r\alpha = \frac{gr^2 \sin \theta}{\kappa^2 + r^2}.$$

Since $\kappa^2 = \frac{r^2}{2}$, we have

$$\alpha = - \frac{2g \sin \theta}{3r}, \; \bar{f}_{tx} = \frac{2g \sin \theta}{3}.$$

Both linear tangential and angular accelerations are constant and the velocity after any time may readily be determined.

(3) *Find the time a rigid homogeneous cylinder will take to roll from rest down a plane 20 ft. long and inclined 30° to the horizon, the axis of the cylinder being horizontal.*

Ans. 1.93 sec.

(4) *A rigid homogeneous circular disk of mass m and radius r, whose plane is vertical, moves in contact with a smooth inclined plane whose angle is θ. From a point in the same vertical plane as the disk and at a distance from the inclined plane equal to the diameter of the disk a string is carried parallel to the inclined plane and is wrapped round the edge of the disk, and its end is fixed in the circumference. Find the tension T in the string, the linear acceleration $\bar{f}$ of the centre, and the angular acceleration α of the disk.*

Ans. $T = \frac{mg\kappa^2 \sin \theta}{\kappa^2 + r^2} = \frac{mg \sin \theta}{3}$ poundals or $\frac{m \sin \theta}{3}$ lbs.;

$$\bar{f} = \frac{gr^2 \sin \theta}{\kappa^2 + r^2} = \frac{2g \sin \theta}{3} \text{ ft.-per-sec. per sec.;}$$

$$\alpha = \frac{gr \sin \theta}{\kappa^2 + r^2} = \frac{2g \sin \theta}{3r} \text{ radians-per-sec. per sec.}$$

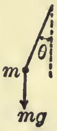
(5) *A perfectly flexible and inextensible ribbon is coiled on the circumference of a homogeneous circular disk of radius r and mass m, and has its free end attached at a fixed point. A part of the ribbon is unrolled and vertical, and the disk is allowed to fall from rest by its own weight. Find the acceleration $\bar{f}$ of the centre and the angular acceleration α before the ribbon becomes wholly unrolled, and the distance s which the centre will descend in one second.*

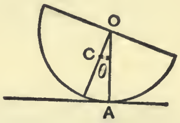
Ans. $\alpha = \frac{2g}{3r}$, $\bar{f} = \frac{2g}{3}$, $s = \frac{1}{2}ft^2 = \frac{g}{3}$ft.

(6) *A homogeneous hemisphere of radius r performs small oscillations on a perfectly rough horizontal plane. Find the periodic time.*

Ans. For the simple pendulum of length l and mass m we have

$$mg \times l \sin\theta = ml^2\alpha, \quad \text{or} \quad \alpha = \frac{g\sin\theta}{l}. \quad (1)$$





For the hemisphere let d be the distance OC from the centre of the hemisphere O to the centre of mass C, and let κ be the radius of gyration about an axis through the centre of mass C parallel to the instantaneous axis at A. Then the moment of inertia for the instantaneous axis at A is

$$I' = m[\kappa^2 + (d\sin\theta)^2 + (r - d\cos\theta)^2].$$

If θ is small we may put $\sin^2\theta = 0$ and $\cos\theta = 1$, and we have

$$I' = m[\kappa^2 + (r-d)^2].$$

We have then

$$I'\alpha = mg \times d\sin\theta, \quad \text{or} \quad \alpha = \frac{gd\sin\theta}{\kappa^2 + (r-d)^2}. \quad . \quad . \quad . \quad . \quad (2)$$

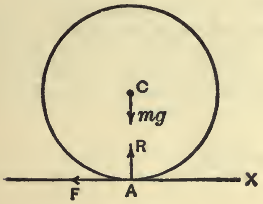
Equating (1) and (2), we have for the length of the equivalent simple pendulum

$$l = \frac{\kappa^2 + (r-d)^2}{d}.$$

The periodic time is then

$$t = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{\kappa^2 + (r-d)^2}{dg}}.$$

(7) *A homogeneous circular hoop moving in a vertical plane in contact with a rough horizontal surface has at a given instant an angular velocity opposite in direction to that which would enable it to roll in the direction of its translation at that instant. Determine its motion.*



Ans. Let m be the mass of the hoop. The forces acting on the hoop are its weight mg at the centre C, the upward pressure of the plane R at A, and the friction F opposite to the direction of translation AX.

The acceleration of the centre is then

$$\bar{f} = -\frac{F}{m},$$

and the angular acceleration upon the axis through C perpendicular to the plane of the hoop is, if κ is the radius of gyration for this axis,

$$\alpha = -\frac{Fr}{m\kappa^2}.$$

We have also

$$R - mg = 0, \quad \text{or} \quad R = mg.$$

If μ is the coefficient of sliding friction we have

$$F = \mu mg.$$

Hence

$$\bar{f} = -\mu g, \quad \alpha = -\frac{\mu g r}{\kappa^2}.$$

If $\bar{v}_1$ and ω_1 are the initial values of the linear and angular velocities, we have then for the linear and angular velocity after any time t

$$\bar{v} = \bar{v}_1 - \mu g t, \quad \omega = \omega_1 - \frac{\mu g r t}{\kappa^2}.$$

If at any instant there is no slipping, we have at that instant the velocity at A zero, or, from page 259,

$$\bar{v} + r\omega = 0.$$

If we eliminate v and ω by means of this equation, we have then for the time after which slipping ceases

$$t = \frac{\kappa^2(\bar{v}_1 + r\omega_1)}{\mu g(\kappa^2 + r^2)}.$$

At this instant there is no tendency to slip and μ becomes zero, and hence at this instant $\bar{f} = 0$ and $\alpha = 0$. Hence after the time t the linear and angular velocities are constant and given by

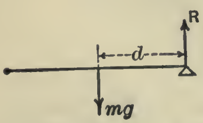
$$\bar{v} = \bar{v}_1 - \frac{\kappa^2(\bar{v}_1 + r\omega_1)}{\kappa^2 + r^2} = \frac{r(r\bar{v}_1 - \kappa^2\omega)}{\kappa^2 + r^2};$$

$$\omega = \omega_1 - \frac{r(\bar{v}_1 + r\omega_1)}{\kappa^2 + r^2} = \frac{\kappa^2\omega - r\bar{v}_1}{\kappa^2 + r^2}.$$

If $r\bar{v}_1 - \kappa^2\omega$ is negative $\bar{v}$ is negative. Hence if ω_1 is positive and greater than $\frac{r\bar{v}_1}{\kappa^2}$, the translation of the hoop after the time t will be in the opposite direction to the initial translation.

(8) *A homogeneous beam is supported horizontally on two supports. Find where one of them must be placed in order that when the other is removed the instantaneous force exerted on the former may be equal to half the weight of the beam.*

Ans. Let d be the required distance of the permanent support from the centre of the beam, κ the radius of gyration of the beam about a normal axis through the centre, m the mass of the beam, α its angular acceleration, and R the reaction of the permanent support immediately after the removal of the other. Then



$$R = \frac{1}{2}mg,$$

and for the centre of the beam

$$m\kappa^2\alpha = Rd, \quad \text{or} \quad \alpha = \frac{Rd}{m\kappa^2} = \frac{gd}{2\kappa^2}.$$

For the end of the beam

$$m(\kappa^2 + d^2)\alpha = mgd, \quad \text{or} \quad \alpha = \frac{gd}{\kappa^2 + d^2}.$$

Equating these two values of α, we have $d = \kappa$.

(9) *A homogeneous circular cylinder of radius r, radius of gyration about the axis* κ, *rotating about its axis with angular velocity* ω_1, *is placed with its axis horizontal on a rough inclined plane (coefficient of friction* μ, *inclination* θ, *so that* $\mu = \tan\theta$), *the direction of rotation being that which it would have if the cylinder were rolling without sliding up the plane. Show that the axis of the cylinder will be stationary for a time* $t = \frac{\kappa^2\omega_1}{\mu r g \cos\theta}$, *at the end of which the angular velocity will be zero.*

(10) *A uniform square is supported in a vertical plane with one diagonal horizontal by two supports, one at each extremity of the diagonal. Show that the initial force on one support when the other is removed is equal to one fourth of the weight of the square.*

(11) *A uniform horizontal bar, suspended from any two points in its length by two parallel cords, is at rest. If one of the cords be cut find the initial tension in the other.*

Ans. $T = \frac{Wl^2}{l^2 + 12d^2}$, where l is the length of the bar, d the distance from its centre of mass to the point of attachment of the uncut cord, and W is the weight of the bar.

(12) *A uniform beam of weight W rests with one end against a smooth vertical wall and the other on a smooth horizontal plane, the inclination to the horizon being* θ. *It is prevented from falling by a string attached to its lower end and to the wall. Find the force between the upper end and the wall when the string is cut.*

Ans. $\frac{1}{2} W \cot\theta$.

(13) *A sphere is laid upon a rough inclined plane of inclination* θ. *Show that it will not slide if the coefficient of friction is equal to or greater than* $\frac{2}{7}\tan\theta$.

(14) *A sphere of radius r whose centre of mass is not at its centre of figure is placed on a rough horizontal plane, coefficient of friction* μ. *Find whether it will slide or roll.*

Ans. Let κ' be the radius of gyration of the sphere about the line through the point of contact at right angles to the plane of the centres of figure and mass. Then if the initial distance of the centre of mass from a vertical through the centre of figure is greater than $\frac{\mu\kappa'^2}{r}$, it will begin to slide; if less, to roll.

CHAPTER IV.

MOTION IN THREE DIMENSIONS.

MOTION OF A BODY IN GENERAL. GENERAL FORMULAS FOR MOTION IN THREE DIMENSIONS. PERMANENT AXIS OF ROTATION. SPONTANEOUS AXIS OF ROTATION. INSTANTANEOUS AXIS OF ROTATION. EQUIVALENT SCREW. CONSERVATION OF MOMENT OF MOMENTUM. INVARIABLE AXIS AND PLANE. MOTION OF A RIGID BODY, NO IMPRESSED FORCES. MOTION OF A RIGID BODY, IMPRESSED FORCES.

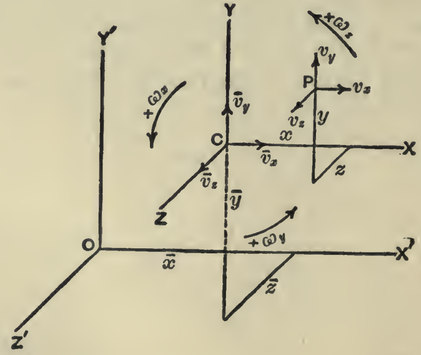
Motion of a Body in General.—The motion of a body in general consists of rotation about an axis through the centre of mass and of translation of the centre of mass.

So far as motion of translation is concerned we can treat the body as a particle at the centre of mass and consider all forces acting upon the body as acting upon this particle without change in magnitude or direction.

So far as motion of rotation is concerned we may consider the centre of mass as fixed.

General Formulas for Motion in Three Dimensions.—Let a rigid body of mass M rotate about an axis through the centre of mass C with the angular velocity ω and the angular acceleration α, and let the components along the co-ordinate axes CX, CY, CZ be ω_x, ω_y, ω_z, α_x, α_y, α_z, positive direction of rotation being counter-clockwise as shown in the figure.

Take a fixed origin O in space and take the co-ordinate axes OX', OY', OZ' parallel to CX, CY, CZ.



Let the co-ordinates of C with reference to O be $\bar{x}$, $\bar{y}$, $\bar{z}$ and the co-ordinates of any point P of the body in general with reference to O be x', y', z', and with reference to C, x, y, z.

In the same way let the components of the velocity of C with reference to O be $\overline{v_x}$, $\overline{v_y}$, $\overline{v_z}$ and the components of the velocity of any point P of the body in general with reference to O be v_x', v_y', v_z', and with reference to C, v_x, v_y, v_z.

So also let the components of the acceleration of C with reference to O be $\overline{f_x}$, $\overline{f_y}$, $\overline{f_z}$, and the components of the acceleration of any point P of the body in general with reference to O be f_x', f_y', f_z', and with reference to C, f_x, f_y, f_z.

Then we have by reason of our notation

$$x' = \overline{x} + x, \qquad y' = \overline{y} + y, \qquad z = \overline{z} + z.$$

For the components v_x, v_y, v_z of P due to rotation about the axis through C we have

$$\left.\begin{aligned}
\frac{dx}{dt} &= v_x = z\omega_y - y\omega_z; \\
\frac{dy}{dt} &= v_y = x\omega_z - z\omega_x; \\
\frac{dz}{dt} &= v_z = y\omega_x - x\omega_y.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

If in these equations we make $\omega_x = 0$, $\omega_y = 0$, we have equations (1a), page 254, for motion in two dimensions.

At the centre of mass C there is no velocity due to rotation, but only velocity of translation. Every point of the body not in the axis of rotation has this velocity of translation and also a velocity of rotation. We have then for the combined velocity of any point P of the body with reference to O

$$\left.\begin{aligned}
\frac{dx'}{dt} &= v_x' = \overline{v_x} + v_x = \overline{v_x} + z\omega_y - y\omega_z; \\
\frac{dy'}{dt} &= v_y' = \overline{v_y} + v_y = \overline{v_y} + x\omega_z - z\omega_x; \\
\frac{dz'}{dt} &= v_z' = \overline{v_z} + v_z = \overline{v_z} + y\omega_x - x\omega_y.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

If in these equations we make $\omega_x = 0$, $\omega_y = 0$, we have equations (1), page 254, for motion in two dimensions. If in addition we make $\overline{v_x} = 0$, $\overline{v_y} = 0$, $y = y'$, $x = x'$, we have equations (1), page 190, for rotation only about a fixed axis.

We have for the components of the tangential acceleration of P with reference to C, due to rotation,

$$f_{tx} = z\alpha_y - y\alpha_z, \qquad f_{ty} = x\alpha_z - z\alpha_x, \qquad f_{tz} = y\alpha_x - x\alpha_y,$$

and for the components of the normal acceleration of P with reference to C, due to rotation,

$$f_{nx} = v_z\omega_y - v_y\omega_z, \qquad f_{ny} = v_x\omega_z - v_z\omega_x, \qquad f_{nz} = v_y\omega_x - v_x\omega_y.$$

At the centre of mass there is no acceleration due to rotation, but only acceleration of translation. Every point of the body not in the axis of rotation has this acceleration of translation and also an acceleration of rotation. We have then for the combined acceleration of any point P of the body with reference to O

$$\left.\begin{aligned}
\frac{dv_x'}{dt} &= f_x' = \overline{f_x} + f_{nx} + f_{tx} = \overline{f_x} + (v_z\omega_y - v_y\omega_z) + (z\alpha_y - y\alpha_z);\\
\frac{dv_y'}{dt} &= f_y' = \overline{f_y} + f_{ny} + f_{ty} = \overline{f_y} + (v_x\omega_z - v_z\omega_z) + (x\alpha_z - z\alpha_x);\\
\frac{dv_z'}{dt} &= f_z' = \overline{f_z} + f_{nz} + f_{tz} = \overline{f_z} + (v_y\omega_x - v_x\omega_y) + (y_{xn} - x_{xy}).
\end{aligned}\right\} (2a)$$

If we put for v_x, v_y, v_z their values as given by (1), we have, since $\omega^2 = \omega_x{}^2 + \omega_y{}^2 + \omega^2{}_z$, by reduction

$$\left.\begin{aligned}
\frac{dv_x'}{dt} &= f_x' = \overline{f_x} + (x\omega_x + y\omega_y + z\omega_z)\omega_x - x\omega^2 + (z\alpha_y - y\alpha_z);\\
\frac{dv_y'}{dt} &= f_y' = \overline{f_y} + (x\omega_x + y\omega_y + z\omega_z)\omega_y - y\omega^2 + (x\alpha_z - z\alpha_x);\\
\frac{dv_z'}{dt} &= f_z' = \overline{f_z} + (x\omega_x + y\omega_y + z\omega_z)\omega_z - z\omega^2 + (y\alpha_x - x\alpha_y).
\end{aligned}\right\} . \ (2)$$

If in these equations we make $\omega_x = 0$, $\omega_y = 0$, $\omega = \omega_z$ and $\overline{f_z} = 0$, $\alpha_x = 0$, $\alpha_y = 0$, we have equations (2), page 255, for motion in two dimensions. If, in addition, we make $\overline{f_x} = 0$, $\overline{f_y} = 0$ and $y = y'$, $x = x'$, we have equations (2), page 191, for rotation only about a fixed axis.

We can obtain equations (2) directly from (1) by differentiating, since

$$\begin{gathered}
\frac{dv_x'}{dt} = f_x', \quad \frac{dv_y'}{dt} = f_y', \quad \frac{dv_z'}{dt} = f_z'.\\
\frac{dv_x}{dt} = f_x, \quad \frac{dv_y}{dt} = f_y, \quad \frac{dv_z}{dt} = f_z;\\
\frac{d\overline{v_x}}{dt} = \overline{f_x}, \quad \frac{d\overline{v_y}}{dt} = \overline{f_y}, \quad \frac{d\overline{v_z}}{dt} = \overline{f_z};\\
\frac{d\omega_x}{dt} = \alpha_x, \quad \frac{d\omega_y}{dt} = \alpha_y, \quad \frac{d\omega_z}{dt} = \alpha_z;\\
\frac{dx}{dt} = v_x, \quad \frac{dy}{dt} = v_y, \quad \frac{dz}{dt} = v_z.
\end{gathered}$$

Since the axis of rotation passes through the centre of mass C, we have, if m is the mass of a particle,

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0. \quad . \ . \ . \ . \ . \ (3)$$

Also

$$\Sigma m = M, \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (4)$$

and

$$\left.\begin{aligned}
&\Sigma m(x^2 + y^2) = I_z, \quad \Sigma m(y^2 + z^2) = I_x, \quad \Sigma m(x^2 + z^2) = I_y;\\
&\Sigma m(x'^2 + y'^2) = I_z', \quad \Sigma m(y'^2 + z'^2) = I_x', \quad \Sigma m(x'^2 + z'^2) = I_y'.
\end{aligned}\right\} (5)$$

Motion of Centre of Mass.—From (2) we have for the sum of the components of all the effective forces, after reduction by (3) and (4),

$$\Sigma m f_x' = M\bar{f_x}, \quad \Sigma m f_y' = M\bar{f_y}, \quad \Sigma m f_z' = M\bar{f_z}. \quad . \quad . \quad . \quad (6)$$

But by D'Alembert's principle (page 168) the sum of the components of the impressed forces in any direction is equal to the sum of the components of the effective forces in that direction.

Hence, *the centre of mass moves at any instant as if all the mass and impressed forces were collected at the centre of mass.*

Momentum.—From (1) we have for the sum of the components of momentum of all the particles, after reduction by (3) and (4),

$$\Sigma m v_x' = M\bar{v_x}, \quad \Sigma m v_y' = M\bar{v_y}, \quad \Sigma m v_z' = M\bar{v_z}. \quad . \quad . \quad . \quad (7)$$

Hence, *the momentum of the body is the same as for all the mass collected at the centre of mass.*

Moment of Momentum.—Let $\mathfrak{M}'_{mx}$, $\mathfrak{M}'_{my}$, $\mathfrak{M}'_{mz}$ be the sums of the moments of momentum of all the particles about the co-ordinate axes OX', OY', OZ'. Then we have from (1) after reduction by (3), (4) and (5),

$$\left.\begin{aligned}
\mathfrak{M}'_{mx} &= \Sigma m(v_z'y' - v_y'z') = I_x\omega_x - \omega_y\Sigma mxy - \omega_z\Sigma mxz + M\bar{v_z}\,\bar{y} - M\bar{v_y}\,\bar{z}\,;\\
\mathfrak{M}'_{my} &= \Sigma m(v_x'z' - v_z'x') = I_y\omega_y - \omega_x\Sigma mxy - \omega_z\Sigma myz + M\bar{v_x}\,\bar{z} - M\bar{v_z}\,\bar{x}\,;\\
\mathfrak{M}'_{mz} &= \Sigma m(v_y'x' - v_x'y') = I_z\omega_z - \omega_x\Sigma mxz - \omega_y\Sigma myz + M\bar{v_y}\,\bar{x} - M\bar{v_x}\,\bar{y}.
\end{aligned}\right\} \quad (8)$$

The last two terms in these equations give the moment of momentum about OX', OY', OZ' due to translation, the others due to rotation about the axis through the centre of mass C.

If we make $\omega_x = 0$, $\omega_y = 0$, $\bar{v_z} = 0$, $\bar{z} = 0$, equations (8) become equations (8), page 255, for motion in two dimensions.

The resultant moment of momentum is given by

$$\mathfrak{M}'_m = \sqrt{\mathfrak{M}'^2_{mx} + \mathfrak{M}'^2_{my} + \mathfrak{M}'^2_{mz}}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the direction-cosines of the resultant axis of moment of momentum are

$$\cos\theta_x = \frac{\mathfrak{M}'_{mx}}{\mathfrak{M}'_m}, \quad \cos\theta_y = \frac{\mathfrak{M}'_{my}}{\mathfrak{M}'_m}, \quad \cos\theta_z = \frac{\mathfrak{M}'_{mz}}{\mathfrak{M}'_m}. \quad . \quad . \quad (10)$$

If we have rotation only without translation, we have from (1), making $\bar{v_x} = 0$, $\bar{v_y} = 0$, $\bar{v_z} = 0$ and taking x', y', z', in place of x, y, z,

$$v_x' = z'\omega_y - y'\omega_z, \quad v_y' = x'\omega_z - z'\omega_x, \quad v_z' = y'\omega_x - x'\omega_y,$$

and equations (8) become in this case, *for rotation only, without translation,*

$$\left.\begin{aligned}
\mathfrak{M}'_{mx} &= \Sigma m(v_z'y' - v_y'z') = I_x'\omega_x - \omega_y\Sigma mx'y' - \omega_z\Sigma mx'z';\\
\mathfrak{M}'_{my} &= \Sigma m(v_x'z' - v_z'x') = I_y'\omega_y - \omega_x\Sigma mx'y' - \omega_z\Sigma my'z';\\
\mathfrak{M}_{mz} &= \Sigma m(v_y'x' - v_x'y') = I_z'\omega_z - \omega_x\Sigma mx'z' - \omega_y\Sigma my'z'.
\end{aligned}\right\} \quad (8a)$$

If the axes OX', OY', OZ' are principal axes we have

$$\Sigma mx'y' = 0, \quad \Sigma mx'z' = 0, \quad \Sigma my'z' = 0;$$

and in this case, *for rotation only*, we have

$$\mathfrak{M}'_{mx} = I_x'\omega_x, \quad \mathfrak{M}'_{my} = I_y'\omega_y, \quad \mathfrak{M}'_{mz} = I_z'\omega_z.$$

That is, *for rotation only without translation, the moment of momentum about a principal axis is equal to the product of the moment of inertia and angular velocity about that axis.*

The same principle holds good for the axis of rotation through the centre of mass C, *whether it is a principal axis or not.* For let this axis be the axis of z, for instance. Then we have $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$, and from the last of equations (8), making $v_x = 0$, $v_y = 0$, we have for rotation only about the axis of rotation through C

$$\mathfrak{M}_{mz} = I_z\omega_z.$$

Kinetic Energy.—Let v' be the velocity of any particle, and $\bar{v}$ the velocity of the centre of mass C with reference to O, so that

$$v'^2 = v_x'^2 + v_y'^2 + v_z'^2, \quad \bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2.$$

Then we have from (1), after reducing by (3), (4) and (5), for the sum of the kinetic energy of all the particles,

$$E' = \Sigma\frac{1}{2}mv'^2 = \frac{1}{2}M\bar{v}^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 - \omega_x\omega_y\Sigma mxy$$
$$- \omega_x\omega_z\Sigma mxz - \omega_y\omega_z\Sigma myz.$$

Now we have for the direction-cosines of the axis of rotation through C

$$\cos\alpha = \frac{\omega_x}{\omega}, \quad \cos\beta = \frac{\omega_y}{\omega}, \quad \cos\gamma = \frac{\omega_z}{\omega},$$

and hence

$$E' = \frac{1}{2}M\bar{v}^2 + \frac{1}{2}\omega^2(I_x\cos^2\alpha + I_y\cos^2\beta + I_z\cos^2\gamma - 2\Sigma mxy\cos\alpha\cos\beta$$
$$- 2\Sigma mxz\cos\alpha\cos\gamma - 2\Sigma myz\cos\beta\cos\gamma).$$

But (page 220) the quantity in parenthesis is the moment of inertia I for the axis of rotation. Hence

$$E' = \frac{1}{2}M\bar{v}^2 + \frac{1}{2}I\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is the same as equation (11), page 256, for motion in two dimensions. The first term on the right is the kinetic energy of translation for the total mass collected at the centre of mass, and the other term is the kinetic energy of rotation about the axis of rotation through C. If there is no translation, we have $\bar{v} = 0$, and hence for rotation only about an axis through C

$$E = \frac{1}{2}I\omega^2.$$

For rotation only about any axis through O we have

$$E' = \frac{1}{2}I'\omega^2;$$

or if OX', OY', OZ' are principal axes,

$$E' = \frac{1}{2} I_x' \omega_x^2 + \frac{1}{2} I_y' \omega_y^2 + \frac{1}{2} I_z' \omega_z^2 . \quad . \quad . \quad . \quad . \quad . \qquad (11a)$$

Moment of the Effective Forces—Euler's Dynamic Equations.—Let $\mathfrak{M}'_{fx}$, $\mathfrak{M}'_{fy}$, $\mathfrak{M}'_{fz}$ be the sums of the moments of the effective forces about the co-ordinate axes OX', OY', OZ'.

Then we have from (2), after reduction by (3), (4) and (5), if the co-ordinate axes *are principal axes*,

$$\left.\begin{aligned}
\mathfrak{M}'_{fx} &= \Sigma m(f_z'y' - f_y'z') = M\overline{f_z}\,\bar{y} - M\overline{f_y}\,\bar{z} + I_x\alpha_x - (I_y - I_z)\omega_y\omega_z ; \\
\mathfrak{M}'_{fy} &= \Sigma m(f_x'z' - f_zx') = M\overline{f_x}\,\bar{z} - M\overline{f_z}\,\bar{x} + I_y\alpha_y - (I_z - I_x)\omega_z\omega_x ; \\
\mathfrak{M}'_{fz} &= \Sigma m(f_y'x' - f_z'y') = M\overline{f_y}\,\bar{x} - M\overline{f_x}\,\bar{y} + I_z\alpha_z - (I_x - I_y)\omega_x\omega_y .
\end{aligned}\right\} (12)$$

If in these equations we make $\overline{f_z} = 0$, $\bar{z} = 0$, $\alpha_x = 0$, $\alpha_y = 0$, $\omega_x = 0$, $\omega_y = 0$, we have equations (12), page 257, for motion in two dimensions and *principal axes.*

In these equations the first two terms on the right give the moments about CX, CY, CZ for the effective force due to translation of the entire mass collected at the centre of mass.

We have (page 265) for principal axes, reducing by (3) and (5),

$$\Sigma m(f_{tz}y - f_{ty}z) = I_x\alpha_x , \quad \Sigma m(f_{tx}z - f_{tz}x) = I_y\alpha_y ,$$
$$\Sigma m(f_{ty}x - f_{tx}y) = I_z\alpha_z .$$

These terms in equations (12) give then the moments about CX, CY, CZ for the effective tangential forces due to rotation about CZ. We have also (page 265) for principal axes, reducing by (3) and (5),

$$\Sigma m(f_{nz}y - f_{ny}z) = -(I_y - I_z)\omega_y\omega_z , \quad \Sigma m(f_{nx}z - f_{nz}x) = -(I_z - I_x)\omega_z\omega_x ,$$
$$\Sigma m(f_{ny}x - f_{nx}y) = -(I_x - I_y)\omega_x\omega_y .$$

These terms in equations (12) give then the moments about CX, CY, CZ for the effective deflecting forces.

If we have rotation only without translation about an axis through O, we have $\overline{f_x}$, $\overline{f_y}$, $\overline{f_z}$ zero in (12); and since in equations (2) x, y, z become x', y', z', equations (12) become, if I_x', I_y', I_z' are the moments of inertia about the principal axes OX', OY', OZ',

$$\left.\begin{aligned}
\mathfrak{M}'_{fx} &= I_x'\alpha_x - (I_y' - I_z')\omega_y\omega_z ; \\
\mathfrak{M}'_{fy} &= I_y'\alpha_y - (I_z' - I_x')\omega_z\omega_x ; \\
\mathfrak{M}'_{fz} &= I_z'\alpha_z - (I_x' - I_y')\omega_x\omega_y .
\end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (12a)$$

These equations are known as *Euler's dynamic equations* for rotation.

Impressed Forces.—Let the impressed forces be F_1, F_2, F_3, etc., making the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc., with the co-ordinate axes. Then we have for the resultant components of the external forces

$$\left.\begin{aligned}
F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \text{etc.} = \Sigma F \cos \alpha ; \\
F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \text{etc.} = \Sigma F \cos \beta ; \\
F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \text{etc.} = \Sigma F \cos \gamma ;
\end{aligned}\right\} \quad . \quad . \qquad (13)$$

Moment of the Impressed Forces.—Let $\mathfrak{M}_{ex}$, $\mathfrak{M}_{ey}$, $\mathfrak{M}_{ez}$ be the sums of the moments of the impressed or external forces about the co-ordinate axes CX, CY, CZ, and let (x_1, y_1, z_1), (x_2, y_2, z_2), etc., be the co-ordinates of the points of application of the impressed or external forces, F_1, F_2, etc. Then we have

$$\left.\begin{aligned} \mathfrak{M}_{ex} &= \Sigma Fy \cos\gamma - \Sigma Fz \cos\beta; \\ \mathfrak{M}_{ey} &= \Sigma Fz \cos\alpha - \Sigma Fx \cos\gamma; \\ \mathfrak{M}_{ez} &= \Sigma Fx \cos\beta - \Sigma Fz \cos\alpha. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (14)$$

Permanent Axis of Rotation.—Let a body rotate about one of the principal axes through the centre of mass, as for instance the axis CZ. Then we have $\omega_x = 0$, $\omega_y = 0$, and from $(12a)$,

$$\mathfrak{M}_{fx} = I_x\alpha_x, \quad \mathfrak{M}_{fy} = I_y\alpha_y, \quad \mathfrak{M}_{fz} = I_z\alpha_z;$$

that is, the moments of the effective deflecting forces about CX, CY, CZ are zero. By D'Alembert's principle we have

$$\mathfrak{M}_{fx} = \mathfrak{M}_{ex}, \quad \mathfrak{M}_{fy} = \mathfrak{M}_{ey}, \quad \mathfrak{M}_{fz} = \mathfrak{M}_{ez}.$$

Now if there are no impressed forces, or if all the impressed forces always pass through the centre of mass, or always form a system whose resultant moment about C is zero, or if the resultant of all the impressed forces always lies in the plane of rotation, we have $\mathfrak{M}_{ex} = \mathfrak{M}_{fx} = 0$, $\mathfrak{M}_{ey} = \mathfrak{M}_{fy} = 0$, or $\alpha_x = 0$, $\alpha_y = 0$, and therefore the axis CZ will remain unchanged in direction and the body will always rotate about it. For this reason the principal axis through the centre of mass in such case is called a **permanent axis** of rotation.

If therefore we observe a body to rotate a short time about an unchanging axis, we infer that it rotated about it from the beginning of the motion and that the axis is a principal axis through the centre of mass. If the angular velocity is uniform, we infer that all the impressed forces are zero or always pass through the centre of mass. If the angular velocity is not uniform we infer that the resultant of all the impressed forces always lies in the plane of rotation.

Spontaneous Axis of Rotation.—The axis through the centre of mass about which a body rotates at any instant we call the **spontaneous axis** of rotation.* If ω is the angular velocity about the spontaneous axis and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are its direction cosines, we evidently have

$$\left.\begin{aligned} \omega &= \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}; \\ \cos\alpha = \frac{\omega_x}{\omega}, \quad \cos\beta &= \frac{\omega_y}{\omega}, \quad \cos\gamma = \frac{\omega_z}{\omega}. \end{aligned}\right\} \quad . \; . \; . \quad (15)$$

Also, if in $(1a)$ we make $v_x = 0$, $v_y = 0$, $v_z = 0$, we obtain the equations of the locus of all those points which have no velocity due to rotation, that is, the equations of the spontaneous axis. Taking then x'', y'', z'' as the co-ordinates for any point of the spontaneous axis with reference to C, we have

$$z'' = \frac{\omega_z}{\omega_y}y'', \quad x'' = \frac{\omega_x}{\omega_z}z'', \quad y'' = \frac{\omega_y}{\omega_x}x''. \quad . \; . \; . \; . \; . \; . \quad (16)$$

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* Usually called the "instantaneous axis." We prefer the term *spontaneous* axis.

These then are the equations of the projections of the spontaneous axis on the co-ordinate planes YZ, ZX, XY, respectively.

We have just seen that if the spontaneous axis is a principal axis and there are no impressed forces, or if all the impressed forces always pass through the centre of mass, or always form a system whose resultant moment about C is zero, or if the resultant of all the impressed forces always lies in the plane of rotation, the direction of the spontaneous axis will remain unchanged and it is a permanent axis of rotation.

If the spontaneous axis is not a principal axis, or if, being a principal axis, the preceding conditions are not fulfilled, it will continually change in direction and describe a cone whose vertex is the centre of mass.

Instantaneous Axis of Rotation.—The axis *fixed in space* about which a body rotates at any instant when the centre of mass *moves in a plane at right angles to the spontaneous axis* we call the **instantaneous axis** of rotation.*

If in equations (1) we make $v_x' = 0$, $v_y' = 0$, $v_z' = 0$, we obtain the equations of the locus of all those points whose velocity is zero at the instant, that is, the equations of the instantaneous axis. Taking then x'', y'', z'' as the co-ordinates for any point of the instantaneous axis *with reference to C*, we have

$$z'' = \frac{\omega_z}{\omega_y}y'' - \frac{\overline{v_x}}{\omega_y}, \quad x'' = \frac{\omega_x}{\omega_z}z'' - \frac{\overline{v_y}}{\omega_z}, \quad y'' = \frac{\omega_y}{\omega_x}x'' - \frac{\overline{v_z}}{\omega_x}. \quad . \qquad (17)$$

These then are the equations of the projections of the instantaneous axis on the co-ordinate planes. Comparing with equations (16), we see that the *instantaneous and spontaneous axes are parallel*, and the velocity of the centre of mass is at right angles to their plane.

If in (17) we make $\omega_x = 0$, $\omega_y = 0$, $\overline{v_z} = 0$, we obtain equations (17), page 259, for motion in two dimensions only.

If the axis of angular acceleration through the centre of mass *coincides with the spontaneous axis*, we can evidently replace ω_x, ω_y, ω_z by α_x, α_y, α_z, and v_x, v_y, v_z by $\overline{f_{tx}}$, $\overline{f_{ty}}$, $\overline{f_{tz}}$, and obtain

$$z'' = \frac{\alpha_z}{\alpha_y}y'' - \frac{\overline{f_{tx}}}{\alpha_y}, \quad x'' = \frac{\alpha_x}{\alpha_z}z'' - \frac{\overline{f_{ty}}}{\alpha_z}, \quad y'' = \frac{\alpha_y}{\alpha_x}x'' - \frac{\overline{f_{tz}}}{\alpha_x}. \qquad (18)$$

If in these equations we make $\alpha_x = 0$, $\alpha_y = 0$, $\overline{f_{tz}} = 0$, we obtain equations (18), page 259, for motion in two dimensions only.

Equivalent Screw. (Compare Vol. I., *Kinematics*, page 201.)—When there is an instantaneous axis it must then be parallel to the spontaneous axis and the motion of the centre of mass must be at right angles to their plane.

But the velocity of the centre of mass in general may make an angle greater or less than 90° with the spontaneous axis. In such case the velocity of the centre of mass may be resolved into a component along the spontaneous axis and at right angles to it, and the entire motion of the body at any instant is then in general equivalent to its rotation about an axis parallel to the spontaneous axis and translation along this axis.

The motion of the body is then a screw motion consisting at any instant of rotation about an axis and translation along it.

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* Usually called the "spontaneous axis." We prefer the term *instantaneous* axis.

We have for the angular velocity about the screw axis, and for its direction-cosines, since it is parallel to the spontaneous axis, equations (15),

$$\left.\begin{array}{c}\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2};\\ \cos\alpha = \dfrac{\omega_x}{\omega}, = \cos\beta = \dfrac{\omega_y}{\omega}, \qquad \cos\gamma = \dfrac{\omega_z}{\omega}.\end{array}\right\} \quad . \; . \; . \quad (19)$$

Let u be the velocity along the screw axis, and other notation as in figure, page 264. Then we have

$$u = \bar{v}_x\frac{\omega_x}{\omega} + \bar{v}_y\frac{\omega_y}{\omega} + \bar{v}_z\frac{\omega_z}{\omega};$$

$$u\frac{\omega_x}{\omega} = v_x', \qquad u\frac{\omega_y}{\omega} = v_y', \qquad u\frac{\omega_z}{\omega} = v_z'.$$

Inserting the values of v_x', v_y', v_z' from (1) and letting x'', y'', z'' be the co-ordinates for any point of the axis of the screw with reference to the centre of mass C, we have from these two equations

$$\frac{u}{\omega} = \frac{\bar{v}_x\omega_x + \bar{v}_y\omega_y + \bar{v}_z\omega_z}{\omega_x^2 + \omega_y^2 + \omega_z^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

$$\left.\begin{array}{c} z'' = \dfrac{\omega_z}{\omega_y}y'' - \dfrac{\bar{v}_x}{\omega_y} + \dfrac{u\omega_x}{\omega\omega_y}, \qquad x'' = \dfrac{\omega_x}{\omega_z}z'' - \dfrac{\bar{v}_y}{\omega_z} + \dfrac{u\omega_y}{\omega\omega_z},\\ y'' = \dfrac{\omega_y}{\omega_x}x'' - \dfrac{\bar{v}_z}{\omega_x} + \dfrac{u\omega_z}{\omega\omega_x}.\end{array}\right\} . \quad (21)$$

Equation (20) gives the *unit pitch* of the screw, or the distance of advance $\frac{u}{\omega}$ during a rotation of one radian.

Equations (21) are the equations of the projections on the co-ordinate planes of the screw axis, which we see from equations (16) is parallel to the spontaneous axis. When there is no translation along the screw axis, that is, when the velocity of the centre of mass is at right angles to the spontaneous axis, we have $u = 0$, and equations (21) become the same as equations (17), and the screw axis is the instantaneous axis.

But when $u = 0$ we have, from (9),

$$\bar{v}_x\omega_x + \bar{v}_y\omega_y + \bar{v}_z\omega_z = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

Equation (22) is then the *condition for rotation only* without translation. If it is satisfied, we have at the instant rotation only about the instantaneous axis given by (17). If it is not satisfied, we have screw motion, or translation along and rotation about the screw axis given by (21).

If we substitute in (21) the value of $\frac{u}{\omega}$ from (20) and reduce, we have for the equations of the axis of the screw

$$\frac{1}{\omega_x}\left(x'' - \frac{\bar{v}_z\omega_y - \bar{v}_y\omega_z}{\omega^2}\right) = \frac{1}{\omega_y}\left(y'' - \frac{\bar{v}_x\omega_z - \bar{v}_z\omega_x}{\omega^2}\right)$$
$$= \frac{1}{\omega_z}\left(z'' - \frac{\bar{v}_y\omega_x - \bar{v}_x\omega_y}{\omega^2}\right).$$

We see from these equations that the central axis of the screw passes through a point whose co-ordinates are

$$x'' = \frac{\overline{v_z}\omega_y - \overline{v_y}\omega_z}{\omega^2}, \quad y'' = \frac{\overline{v_x}\omega_z - \overline{v_z}\omega_x}{\omega^2}, \quad z'' = \frac{\overline{v_y}\omega_x - \overline{v_x}\omega_y}{\omega^2}. \quad . \quad (23)$$

If $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$, and $\overline{v_z} = 0$, these equations reduce to equations (17), page 259, for motion in two dimensions.

If we substitute these values of x'', y'', z'' in (21) we have

$$\left.\begin{aligned} \overline{v_x} &= \frac{u}{\omega}\omega_x - (z''\omega_y - y''\omega_z); \\ \overline{v_y} &= \frac{u}{\omega}\omega_y - (x''\omega_z - z''\omega_x); \\ \overline{v_z} &= \frac{u}{\omega}\omega_z - (y''\omega_x - x''\omega_y). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The motion of the body at any instant is therefore known when the six quantities, $\overline{v_x}$, $\overline{v_y}$, $\overline{v_z}$, ω_x, ω_y, ω_z, are known. These six quantities are therefore called the *components of motion.*

If they are known we can find ω and the direction of the axis of the screw from (19), the position with reference to the centre of mass from (23) and the velocity of advance along the central axis from (20).

On the other hand, if the position of the central axis of the screw (x'', y'', z'') is given, together with the velocity u along it, the rotation about it and its direction-cosines, the components $\overline{v_x}$, $\overline{v_y}$, $\overline{v_z}$ are given by (24).

If the spontaneous axis is a principal axis through the centre of mass and there are no impressed forces, or if all the impressed forces pass through the centre of mass, then, as we have seen (page 270), the spontaneous axis is a permanent axis of rotation and the parallel screw axis is a permanent axis also.

If the spontaneous axis is not a principal axis, then even if there are no impressed forces it changes in direction by reason of the moment of the deflecting forces, and the parallel screw axis therefore also changes its direction continually.

Conservation of Moment of Momentum.—We have from (8) for the sums of the moments of momentum of all the particles about the co-ordinate axes OX', OY', OZ',

$$\left.\begin{aligned} \mathfrak{M}'_{mx} &= \Sigma m(v_z'y' - v_y'z'); \\ \mathfrak{M}'_{my} &= \Sigma m(v_x'z' - v_z'x'); \\ \mathfrak{M}'_{mz} &= \Sigma m(v_y'x' - v_x'y'). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

We also have from (12), for the sums of the moments of the effective forces about these axes,

$$\left.\begin{aligned} \mathfrak{M}'_{fx} &= \Sigma m(f_z'y' - f_y'z'); \\ \mathfrak{M}'_{fy} &= \Sigma m(f_x'z' - f_z'x'); \\ \mathfrak{M}'_{fz} &= \Sigma m(f_y'x' - f_x'y'). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

We can obtain these last equations (*b*) directly from the preceding (*a*) by differentiating (*a*), since

$$\frac{dx'}{dt} = v_x', \quad \frac{dy'}{dt} = v_y', \quad \frac{dz'}{dt} = v_z';$$

$$\frac{dv_x'}{dt} = f_x', \quad \frac{dv_y'}{dt} = f_y', \quad \frac{dv_z'}{dt} = f_z'.$$

Hence the integration of equations (*b*) will give us equations (*a*).

Now by D'Alembert's principle the moment of the effective forces is equal to the moment of the impressed forces. But if the moment of all the impressed forces about any axis, as for instance the axis of OZ', is always zero, we have always $\mathfrak{M}'_{fz} = 0$, and hence by integration

$$\mathfrak{M}'_{mz} = C_z,$$

where C_z is a *constant* of integration.

Hence, *if the moment of the impressed forces about any axis is always zero, the moment of momentum about that axis is constant.*

This is the principle of *conservation of momentum.*

Invariable Axis and Plane.—If all the impressed forces are zero, or if their resultant passes always through the centre of mass, we have always

$$\mathfrak{M}_{fx} = 0, \qquad \mathfrak{M}_{fy} = 0, \qquad \mathfrak{M}_{fz} = 0,$$

and hence by integration

$$\mathfrak{M}_{mx} = C_x, \qquad \mathfrak{M}_{my} = C_y, \qquad \mathfrak{M}_{mz} = C_z,$$

where C_x, C_y, C_z are *constants* of integration. The resultant moment of momentum

$$\mathfrak{M}_m = \sqrt{C_x^2 + C_y^2 + C_z^2}$$

is then constant and its projection *in any fixed direction in space is constant.*

Hence, *when a body or system of bodies is acted upon by mutual actions of the particles only, or by any system of impressed forces for which the resultant always passes through the centre of mass, the resultant moment of momentum is constant and the direction of the axis of the resultant moment of momentum is fixed in space.*

The axis of the resultant moment of momentum through the centre of mass in such case is called the **invariable axis**, and the plane through the centre of mass at right angles to this axis is the **invariable plane.**

The invariable plane for a system of bodies is then that plane through the centre of mass on which the sum of the projections of the moment of momentum of all the bodies is a minimum.

If ω is the angular velocity about the spontaneous axis at any instant and I is the moment of inertia of a body about the spontaneous axis at that instant, then (page 171)

$$I\omega$$

is the moment of momentum about the spontaneous axis.

If we take the principal axes through the centre of mass as co-ordinate axes, we have

$$\mathfrak{M}_{mx} = I_x\omega_x, \qquad \mathfrak{M}_{my} = I_y\alpha_y, \qquad \mathfrak{M}_{mz} = I_z\omega_z,$$

and hence for the moment of momentum about the invariable axis

$$\mathfrak{M}_m = \sqrt{I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2}. \quad . \quad . \quad . \quad . \quad (25)$$

The direction-cosines of the invariable axis are then

$$\cos\theta_x = \frac{I_x\omega_x}{\mathfrak{M}_m}, \qquad \cos\theta_y = \frac{I_y\omega_y}{\mathfrak{M}_m}, \qquad \cos\theta_z = \frac{I_z\omega_z}{\mathfrak{M}_m}. \quad . \quad . \quad (26)$$

The angle ϕ of the invariable axis with the spontaneous axis is given by

$$\cos\phi = \frac{I\omega}{\mathfrak{M}_m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The angular velocity about the invariable axis is then

$$\varpi = \omega_x\cos\theta_x + \omega_y\cos\theta_y + \omega_z\cos\theta_z = \omega\cos\phi,$$

or, from (27),

$$\varpi = \frac{I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2}{\mathfrak{M}_m} = \frac{I\omega^2}{\mathfrak{M}_m} = \frac{2E}{\mathfrak{M}_m}. \quad . \quad . \quad . \quad (28)$$

where (page 268) $E = \frac{1}{2}I\omega^2$ is the kinetic energy of rotation about the spontaneous axis.

The velocity of translation along the invariable axis, if u is the velocity of translation along the spontaneous axis, is

$$U = u\cos\phi,$$

or, from equations (20) and (27),

$$U = \frac{I}{\mathfrak{M}_m}(\bar{v}_x\omega_x + \bar{v}_y\omega_y + \bar{v}_z\omega_z). \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Motion of a Rigid Body—No Impressed Forces.—Let the impressed forces be zero and let the body have the angular velocity ω at any instant about the spontaneous axis. Then the centre of mass at any instant has a velocity of translation u along and of rotation ω about the screw axis as given page 272. It only remains to discuss the rotation of the body.

The kinetic energy $E = \frac{1}{2}I\omega^2 = \frac{1}{2}\Sigma mv^2$ is constant, and we have from (28)

$$\Sigma mv^2 = I\omega^2 = \frac{M\rho^4\omega^2}{l^2} = \mathfrak{M}_m\omega\cos\phi, \quad . \quad . \quad . \quad . \quad (30)$$

where (page 220) $l = \frac{\rho^2}{\kappa}$ (where ρ is any arbitrary distance and κ is the radius of gyration for the spontaneous axis) is the semi-diameter of the central ellipsoid of inertia which coincides with the spontaneous axis.

We have from (30)

$$\omega = \frac{l}{\rho^2}\sqrt{\frac{\Sigma mv^2}{M}} = \sqrt{\frac{\Sigma mv^2}{I}}.$$

Hence we conclude, since Σmv^2 is constant,

(1) The angular velocity ω about the spontaneous axis is directly proportional to the length l of the semi-diameter of the central ellipsoid of inertia which coincides with the spontaneous axis, or inversely as the square root of the moment of inertia with respect to that axis.

We have also from (30)

$$\frac{I\omega^2}{\mathfrak{M}m} = \frac{2E}{\mathfrak{M}m} = \omega \cos \phi.$$

Hence we conclude from (28),

(2) The angular velocity about the invariable axis is constant. Therefore as ϕ increases or $\cos \phi$ decreases, ω increases. That is, as the inclination ϕ of the spontaneous axis to the invariable axis increases, the angular velocity ω about the spontaneous axis increases.

The equation of the invariable plane is from (26) since it passes through the centre of mass at right angles to the invariable axis

$$x \cos \theta_x + y \cos \theta_y + z \cos \theta_z = 0,$$

or

$$\frac{I_x\omega_x}{\mathfrak{M}m}x + \frac{I_y\omega_y}{\mathfrak{M}m}y + \frac{I_z\omega_z}{\mathfrak{M}m}z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Call the point in which the spontaneous axis pierces the central ellipsoid of inertia the *spontaneous pole*, and let x', y', z' be its co-ordinates, and l_x, l_y, l_z the principal semi-diameters. Then the equation of the tangent plane to the ellipsoid at the spontaneous pole is

$$\frac{xx'}{l_x^2} + \frac{yy'}{l_y^2} + \frac{zz'}{l_z^2} = 1.$$

But from (19)

$$l\frac{\omega_x}{\omega} = x', \qquad l\frac{\omega_y}{\omega} = y', \qquad l\frac{\omega_z}{\omega} = z'.$$

Hence the equation of the tangent plane to the ellipsoid at the spontaneous pole reduces to

$$\frac{\omega_x x}{l_x^2} + \frac{\omega_y y}{l_y^2} + \frac{\omega_z z}{l_z^2} = \frac{\omega}{l}.$$

Now (page 220)

$$l_x^2 = \frac{\rho^4}{\kappa_x^2}, \quad l_y^2 = \frac{\rho^4}{\kappa_y^2}, \quad l_z^2 = \frac{\rho^4}{\kappa_z^2}, \quad l = \frac{\rho^2}{\kappa}.$$

Hence we have

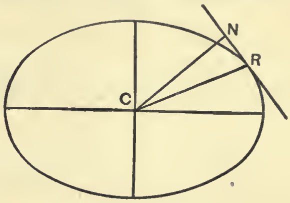
$$\omega_x x \kappa_x^2 + \omega_y y \kappa_y^2 + \omega_z z \kappa_z^2 = \omega\kappa^2 l.$$

Multiplied by M and dividing by $\mathfrak{M}m$, we have finally for the equation of the tangent plane to the ellipsoid at the spontaneous pole

$$\frac{I_x\omega_x}{\mathfrak{M}m}x + \frac{I_y\omega_y}{\mathfrak{M}m}y + \frac{I_z\omega_z}{\mathfrak{M}m}z = \frac{I\omega l}{\mathfrak{M}m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Comparing with (31), we see that the tangent plane at the spontaneous pole is parallel to the invariable plane, and that these two planes are separated by the perpendicular distance (equation (27)) $l \cos \phi = \frac{I\omega}{\mathfrak{M}m}l.$

Thus if CR coincides with the spontaneous axis, and R is the spontaneous pole, the tangent plane RN at R is parallel to the invariable plane and the normal CN is $l \cos \phi = \frac{I\omega}{\mathfrak{M}m} l$, and CN coincides with the invariable axis.



Hence the ellipsoid rotates about CN and rolls without sliding on the tangent plane NR parallel to the invariable plane at the fixed distance $l \cos \phi = \frac{I\omega}{\mathfrak{M}m} l$.

As different points of the ellipsoid come successively into the tangent plane, the semi-diameters which join them with the centre become in turn the spontaneous axis.

The motion of the body at any instant is a screw motion for the spontaneous axis at that instant, as given page 272. The angular velocity about and linear velocity along the invariable axis at that instant are given by (28) and (29).

Evidently, if the spontaneous axis coincides with a principal axis, CN coincides with CR, which is a permanent axis (page 270).

Motion of a Rigid Body—Impressed Forces—Euler's Geometric Equations.—By means of Euler's dynamic equations (page 269) we can find ω_x, ω_y, ω_z when the impressed forces are known, by D'Alembert's principle.

But since the principal axes move with the body, the complete solution requires that the position of these axes at any instant shall be determined with respect to the fixed axes.

We need then the geometrical equations between the motion of a body in space and the angular velocity about an axis.

These equations we have already deduced in Vol. I, *Kinematics*, page 221. They are known as *Euler's geometric equations*. We reproduce them here.

Let OX', OY', OZ' be three rectangular principal axes of the body at the point O. These axes are fixed in the body and rotate with it. Let the body rotate about some axis through the point O, also fixed in the body and therefore making invariable angles with these axes, so that the component angular velocities are ω_x', ω_y', ω_z. We take direction of rotation as always

$$\left.\begin{array}{llll} \text{about } X' & \text{from } Y' & \text{to } Z' \\ \text{“} \quad Y' & \text{“} \quad Z' & \text{to } X' \\ \text{“} \quad Z' & \text{“} \quad X' & \text{to } Y' \end{array}\right\} \begin{array}{l} \text{positive,} \\ \text{the opposite} \\ \text{direction} \\ \text{negative.} \end{array}$$

Let now OX, OY, OZ be rectangular co-ordinate axes whose directions *in space* are invariable.

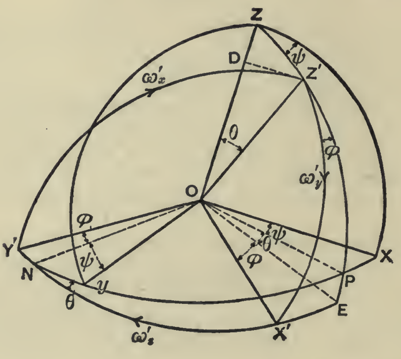
Let O be the centre of a sphere of radius r. Let X, Y, Z and X', Y', Z' be the points in which this sphere is pierced by the fixed and moving axes.

Let the axes OX', OY', OZ' have the initial positions OX, OY, OZ. First let the body rotate about OZ through the angle $XOP = \psi$, so that OX moves to OP and OY to ON. Then rotate the body about ON through the angle $ZOZ' = \theta$, so that OP moves to OE and OZ to OZ'. Finally rotate the body about OZ' through the angle $EOX' = \phi$, so that OE moves to OX' and ON to OY'.

It is required to find the geometric relations between ϕ, θ, ψ and ω_x', ω_y', ω_z'. These geometric relations are called *Euler's geometric equations for rotation.*

The line ON is called the *line of nodes*, θ is the *obliquity*, and ψ the *precession.*

Let the angular velocity of Z' perpendicular to the plane ZOZ' at any instant be $\frac{d\psi}{dt}$. This is called the *angular velocity of precession.* Let the angular velocity of Z' along ZZ' at the same instant be $\frac{d\theta}{dt}$. This is called the *angular velocity of nutation.* Let the angular velocity of X' with reference to E at the instant be $\frac{d\phi}{dt}$.



Draw $Z'D$ perpendicular to OZ. Then $Z'D = r \sin \theta$ and the velocity at any instant of Z' perpendicular to the plane ZOZ' is $r \sin \theta \,.\, \frac{d\psi}{dt}$, and along ZZ' at the same instant it is $r\frac{d\theta}{dt}$. The velocity at the same instant of Z' along $Y'Z'$ is $r\omega_x'$, and along $Z'X'$ it is $r\omega_y'$.

We have then directly from the figure

$$r\frac{d\theta}{dt} = r\omega_y' \cos \phi + r\omega_x' \sin \phi;$$

$$r \sin \theta \,.\, \frac{d\psi}{dt} = r\omega_y'\omega_y \sin \phi - r\omega_x' \cos \phi.$$

Combining these two equations, we obtain

$$\omega_x' = \frac{d\theta}{dt} \sin \phi - \frac{d\psi}{dt} \sin \theta \cos \phi;$$

$$\omega_z' = \frac{d\theta}{dt} \cos \phi + \frac{d\psi}{dt} \sin \theta \sin \phi.$$

In the same way we have the velocity of E perpendicular to the plane ZOE equal to $r \sin (90 + \theta)\frac{d\psi}{dt}$ or $r \cos \theta \,.\, \frac{d\psi}{dt}$, and of X' along EX', relative to E, $r\frac{d\phi}{dt}$. The entire velocity of X' in space along $X'Y'$ is $r\omega_z'$.

Hence

$$r\omega_z' = r\cos\theta \cdot \frac{d\psi}{dt} + r\frac{d\phi}{dt},$$

or

$$\omega_z' = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}.$$

We have then

$$\left.\begin{aligned} \omega_x' &= \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi\,; \\ \omega_y' &= \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi\,; \\ \omega_z' &= \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}\,; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (33)$$

and these are *Euler's geometric equations.*

Auxiliary Angles.—From the spherical triangles of the figure, considering N as a vertex in each, we have for the direction cosines of the moving axis with respect to the fixed,

$$\left.\begin{aligned} \cos X'OX &= -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta, \\ \cos Y'OX &= -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta, \\ \cos Z'OX &= \sin\theta\cos\psi\,; \\[6pt] \cos X'OY &= \cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta, \\ \cos Y'OY &= \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta, \\ \cos Z'OY &= \sin\theta\sin\psi\,; \\[6pt] \cos X'OZ &= -\sin\theta\cos\phi, \\ \cos Y'OZ &= \sin\theta\sin\phi, \\ \cos Z'OZ &= \cos\theta. \end{aligned}\right\} \quad . \quad . \quad (34)$$

For the angles which the axes Z, Z', and ON make with the axes X', Y', Z', we have

$$\left.\begin{aligned} \cos ZOX' &= -\cos\phi\sin\theta; \\ \cos ZOY' &= \sin\phi\sin\theta; \\ \cos ZOZ' &= \cos\theta; \\[6pt] \cos Z'OX' &= 0; \\ \cos Z'OY' &= 0; \\ \cos Z'OZ' &= 1; \\[6pt] \cos NOX' &= \sin\phi; \\ \cos NOY' &= \cos\phi; \\ \cos NOZ' &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

EXAMPLES.

(1) *A layer of dust of uniform depth d feet, d being small, is formed on the earth by the fall of meteors reaching the earth from all directions. Consider the earth as a homogeneous sphere of radius r and density* Δ, *and let* δ *be the density of the dust. Find the change in length of the day.*

Ans. Let ω_1 be the angular velocity of rotation before, and ω_2 after, the layer is formed, and I_1 the moment of inertia of the earth and I_2 that of the layer.

Since there are no forces in the system except the mutual action of the particles, by the principle of conservation of moment of momentum (page 274), we have

$$I_1\omega_1 = (I_1 + I_2)\omega_2; \quad \text{or,} \quad \omega_2 = \frac{1}{1+\frac{I_2}{I_1}}\omega_1.$$

The mass of the earth is given by

$$M_1 = \frac{4}{3}\Delta\pi r^3;$$

hence (page 176) we have

$$I_1 = \frac{2M_1r^2}{5} = \frac{8}{15}\Delta\pi r^5.$$

The moment of inertia of the dust-layer is then

$$I_2 = \frac{8}{15}\delta\pi(r+d)^5 - \frac{8}{15}\delta\pi r^5;$$

hence we have

$$\frac{I_2}{I_1} = \frac{\delta[(r+d)^5 - r^5]}{\Delta r^5}.$$

Expanding, and neglecting $\frac{d^2}{r^2}$ and all higher powers, we have

$$\frac{I_2}{I_1} = \frac{5\delta d}{\Delta r}.$$

Therefore we have

$$\omega_2 = \frac{1}{1+\frac{5\delta d}{\Delta r}}\omega_1.$$

(2) *In the preceding example, if the density of the dust is twice that of water, and* $d = \frac{1}{20}r$, *find the length of day.*

Ans. The mean density of the earth is about 5.5 that of water. Therefore $\frac{\delta}{\Delta} = \frac{2}{5.5}$. We have then

$$\omega_2 = \frac{\omega_2}{1+\frac{5\times 2}{5.5\times 20}} = \frac{11}{12}\omega_1.$$

The length of day would then be $\frac{11}{12}$ of 24 hours, or only 22 hours.

(3) *A bead of mass m slides on a circular wire of mass M and radius r, and the wire rotates about a vertical diameter. If* ω_1 *and* ω_2 *are the angular velocities of the wire when the bead is respectively at the extremities of a horizontal and vertical diameter, show that* $\frac{\omega_2}{\omega_1} = 1 + 2\frac{m}{M}$.

Ans. Let $I = \frac{Mr^2}{2}$ be the moment of inertia of the wire (page 176) and mr^2

that of the bead. Then, since there are no forces except mutual actions of the particles, we have, by the principle of conservation of moment of momentum (page 274),

$$I\omega_1 + mr^2\omega_1 = I\omega_2\,; \quad \text{or,} \quad \frac{Mr^2}{2}\omega_1 + mr^2\omega_1 = \frac{Mr^2}{2}\omega_2\,;$$

whence we obtain

$$\frac{\omega_2}{\omega_1} = 1 + 2\frac{m}{M}.$$

(4) *If the earth gradually contracted by radiation of heat, so as to be always similar to itself as regards its physical constitution and form, show that when every radius vector has contracted an nth part of its length, where n is small, the angular velocity has increased a 2nth part of its value.*

Ans. Let M be the mass of the earth, r_1 its radius before and r_2 that after contraction, and I_1 the moment of inertia before, and I_2, after. Then

$$I_1 = \frac{2Mr_1^2}{5}, \quad I_2 = \frac{2Mr_2^2}{5},$$

and

$$I_1\omega_1 = I_2\omega_2, \quad \text{or} \quad \omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{r_1^2}{r_2^2}.$$

But $r_2 = r_1 - nr_1 = r_1(1 - n)$. Hence we have

$$\omega_2 = \frac{r_1^2}{r_1^2(1 - n^2)} = \frac{1}{(1 - n)^2}.$$

Expanding, and neglecting n^2 and higher powers,

$$\omega_2 = \frac{1}{1 - 2n}\omega_1 = (1 + 2n)\omega_1.$$

(5) *If two railway trains, each of mass M, were to travel in opposite directions from the pole, along a meridian, and to arrive at the equator at the same time, show that the angular velocity of the earth would be decreased by $\frac{2Mr^2}{E\kappa^2}$ of itself; where r is the equatorial radius, E is the mass of the earth, and κ^2 is radius of gyration for the axis.*

Ans. Let I be the moment of inertia of the earth and ω_1, ω_2 the angular velocities before and after. Then $I = E\kappa^2$, and

$$I\omega_1 = I_1\omega_2 + 2Mr^2\omega_2, \quad \text{or} \quad E\kappa^2\omega_1 = E\kappa^2\omega_2 + 2Mr^2\omega_2\,;$$

hence

$$\omega_2 = \frac{\kappa^2}{\kappa^2 + \dfrac{2Mr^2}{E\kappa^2}}\omega_1 = \left(1 - \frac{2Mr^2}{E\kappa^2}\right)\omega_1.$$

(6) *Suppose a mass of ice M to melt from the polar regions for 20° round each pole to the extent of something more than a foot thick, enough to give $1\frac{1}{10}$ feet over those areas or 0.066 of a foot of water spread over the whole globe, which would raise the sea-level by only some such undiscoverable difference as three fourths of an inch or an inch. Show that this would slacken the earth's rotation by one tenth of a second per year.*

Ans. The moment of inertia of the ice-caps before melting is easily shown to be, if θ is the angle of 20°,

$$-\frac{Mr^2}{3}\cos\theta\,(1 + \cos\theta).$$

The moment of inertia of the earth, if E is the mass of the earth considered as a homogeneous sphere, is $\frac{2Er^2}{5}$. If ω_1 is the angular velocity before and ω_2 after melting, we have

$$\frac{2Er^2}{5}\omega_1 - \frac{Mr^2}{3}\cos\theta(1+\cos\theta)\omega_1 = \frac{2Er^2}{5}\omega_2,$$

or

$$\frac{\omega_1 - \omega_2}{\omega_1} = \frac{5M}{6E}\cos\theta(1+\cos\theta).$$

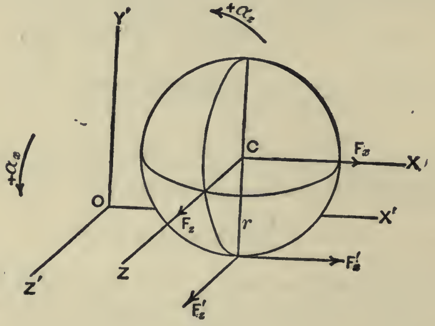
Substituting numerical values, the value of $\omega_1 - \omega_2$ is easily found.

(7) *Find the motion of a sphere rolling on a rough plane.*

Ans. Let the plane be the plane of $X'Z'$, and let the components of the friction parallel to these axes be F_x', F_z'. All the impressed forces can be reduced to a single force F at the centre of mass C and a couple which causes angular acceleration α about an axis through C.

Let F_x, F_z be the components of this force parallel to the axes, and α_x, α_z the components of the angular acceleration.

Let r be the radius of the sphere, κ its radius of gyration about a diameter so that (page 176) $\kappa^2 = \frac{2}{5}r^2$, and let M be the mass of the sphere. Take the axes CX, CY, CZ through the centre of mass C parallel at any instant to OX', OY', OZ'.



Then we have for the moment of the impressed forces about CX and CZ $\mathfrak{M}_{ex} = -F_z'r$, $\mathfrak{M}_{ez} = F_x'r$; and for the moments of the effective forces about the same axes (page 269), if I_x, I_y are the moments of inertia about these axes, $\mathfrak{M}_{fx} = I_x\alpha_x$, $\mathfrak{M}_{fz} = I_z\alpha_z$. We have then by D'Alembert's principle

$$I_x\alpha_x = -F_z'r, \qquad I_z\alpha_z = F_x'r. \quad \dots\dots\dots \quad (1)$$

Let $\bar{f}_x$, $\bar{f}_z$ be the linear accelerations parallel to OX', OZ' of the centre of mass. Then, since the centre of mass moves as if all the forces acted at the centre of mass (page 267), we have, from (1),

$$\left.\begin{aligned} M\bar{f}_x &= F_x + F_x' = F_x + \frac{I_z\alpha_z}{r}; \\ M\bar{f}_z &= F_z + F_z' = F_z - \frac{I_x\alpha_x}{r}. \end{aligned}\right\} \quad \dots\dots \quad (2)$$

But

$$\bar{f}_x = -r\alpha_z, \qquad \bar{f}_z = r\alpha_x, \quad \text{or} \quad \alpha_x = \frac{\bar{f}_z}{r}, \qquad \alpha_z = -\frac{\bar{f}_x}{r}.$$

Substituting in (2), we obtain

$$\overline{f_x} = \frac{r^2}{I_z + Mr^2}F_x, \qquad \overline{f_z} = \frac{r^2}{I_x + Mr^2}F_z.$$

Or, since $I_x = I_z = M\kappa^2$,

$$M\overline{f_x} = \frac{r^2}{\kappa^2 + r^2}F_x, \quad M\overline{f_z} = \frac{r^2}{\kappa^2 + r^2}F_z. \quad . \quad . \quad . \quad . \quad (3)$$

These are the same equations as for a particle of equal mass acted upon by $\frac{r^2}{\kappa^2 + r^2} = \frac{5}{7}$ of the acting forces.

Hence, *the motion of the centre of mass of a homogeneous sphere rolling on a rough plane under the action of any forces is the same as for a particle of the same mass if all the forces are reduced to five sevenths of their former value.*

Now

$$\sqrt{\overline{f_x}^2 + \overline{f_z}^2} = \overline{f} = \text{the resultant horizontal acceleration,}$$

and

$$\sqrt{F_x^2 + F_z^2} = H = \text{the resultant impressed force.}$$

Hence, from (3), we have

$$M\sqrt{\overline{f_x}^2 + \overline{f_z}^2} = \frac{5}{7}\sqrt{F_x^2 + F_z^2}, \quad \text{or} \quad M\overline{f} = \frac{5}{7}H.$$

If F_y is the normal force and μ the coefficient of friction, then μF_y is the friction; and since the centre of mass moves as if all the forces were applied there, we have also

$$M\overline{f} = H - \mu F_y.$$

Hence

$$\frac{5}{7}H = H - \mu F_y, \quad \text{or} \quad \mu = \frac{2}{7}\frac{H}{F_y}.$$

If, then, the coefficient of friction is equal to or greater than $\frac{2}{7}\frac{H}{F_y}$, the sphere *will roll without sliding.*

(8) *A sphere is placed on an inclined plane sufficiently rough to prevent sliding, and a velocity in any direction is communicated to it. Show that the path of the centre is a parabola. If v is the initial horizontal velocity of the centre, and α the inclination of the plane, show that the latus rectum will be* $\frac{14}{5}\frac{v^2}{g \sin \alpha}$.

Ans. The acceleration down the plane, from the principle of the preceding example, is $\frac{5}{7}g \sin \alpha$. If the initial velocity v makes an angle θ with the line of slope, the velocity down the plane is $v \cos \theta$, and at right angles $v \sin \theta$. There is no acceleration at right angles. The distance y passed over at right angles in the time t is then

$$y = vt \sin \theta,$$

and the distance x down the plane is

$$x = vt \cos \theta + \frac{5}{14}gt^2 \sin \alpha.$$

Eliminating t, we have for the equation of the curve

$$x = \frac{y}{\tan\theta} + \frac{5}{14}\frac{g\sin\alpha}{v^2\sin^2\theta}y^2.$$

This is the equation of a parabola. If the initial velocity is horizontal, $\theta = 90°$, $\sin\theta = 1$, $\tan\theta = \infty$, and we have

$$y^2 = \frac{14}{5}\frac{v^2}{g\sin\alpha}x.$$

(9) *A homogeneous sphere rolls on a rough plane under the action of a force varying inversely as the square of the distance from a point in the plane of motion of the centre. Show that its centre describes a conic section. Also, if when the distance of its centre from the centre of force is one quarter of the major axis of its orbit, the sphere comes to a smooth part of the plane, the major axis of the orbit will be suddenly reduced in the ratio* 7 *to* 13.

Ans. We have the central acceleration

$$f = -\frac{a'r'^2}{r^2},$$

where a' is the acceleration at a known distance r'.

Hence (Vol. I, *Kinematics*, page 142)

$$r = \frac{\dfrac{c^2}{a'r'^2}}{1 + \sqrt{1 + \dfrac{c_1c^2}{a'^2r'^4}}\cos(\theta - \phi)},$$

which is the polar equation of a conic section. We have also (Vol. I, *Kinematics*, page 145)

$$v^2 = \frac{a'r'^2}{Ar}(2A - r),$$

where A is the semi-major axis. When $r = \frac{A}{2}$, we have

$$v^2 = \frac{3a'r'^2}{A}.$$

For a smooth plane we have $\frac{7}{5}a'r'^2$ instead of $a'r'^2$, and hence

$$v^2 = \frac{7a'r'^2}{5A_1r}(2A_1 - r).$$

When $r = \frac{A}{2}$, we have

$$v^2 = \frac{28a'r'^2}{5A} - \frac{7a'r'^2}{5A_1} = \frac{3a'r'^2}{A}.$$

Hence $\frac{A_1}{A} = \frac{7}{13}$.

(10) *A homogeneous sphere moves without rotation on a smooth horizontal plane under the action of a central force such that the centre of the sphere describes an ellipse with the centre of force in the focus. If the*

sphere arrive at a part of the plane which is rough when the distance of its centre from the centre of force is $\frac{1}{n}$*th of the major axis of its orbit for the rough plane, show that the major axis is diminished in the ratio 7 to* $5 + 2n$. *If the sphere come again to the smooth part of the plane when the distance of its centre from the focus is the same fraction as before of the major axis of its orbit for the rough plane, show that the major axis is again diminished in the same ratio.*

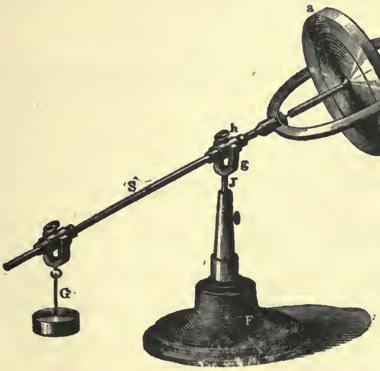
(11) *Show as in example* (7) *that the motion of the centre of mass of a homogeneous disk rolling on a rough plane under the action of any forces is the same as for a particle of the same mass if all the forces are reduced to two thirds of their former value. Also, that the disk will roll without sliding if the coefficient of friction is equal to or greater than* $\frac{1}{3}\frac{H}{F_y}$.

(12) *Show that the motion of the centre of mass of a very thin circular hoop rolling on a rough plane under the action of any forces is the same as for a particle of the same mass if all the forces are reduced to one half their former value. Also that the hoop will roll without sliding if the coefficient of friction is equal to or greater than* $\frac{1}{2}\frac{H}{F_y}$.

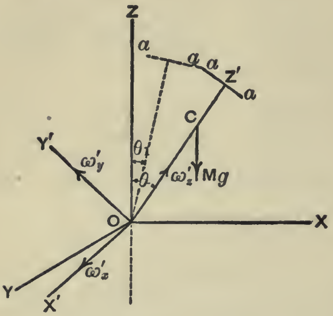
(13) *Show that the motion of the centre of mass of a very thin spherical shell rolling on a rough plane under the action of any forces is the same as for a particle of the same mass if all the forces are reduced to three fifths their former value. Also that the shell will roll without sliding if the coefficient of friction is equal to or greater than* $\frac{2}{5}\frac{H}{F_y}$.

(14) *Discuss the problem of the Gyroscope.*

Description.—The gyroscope consists of a disk *aa* which is set in rotation about an axis in the direction of the diameter of the ring *R*. This ring is attached to the rod *S*, which passes through a sleeve at *h*. This sleeve is pivoted in the fork *g* so that *S* can rotate in a vertical plane, and the fork *g* is pivoted at *J* so that this rod can rotate horizontally. A sliding counterweight *G* can be so adjusted that the centre of mass of the apparatus can be made to lie on the same side of the standard as the disk, on the opposite side, or directly over the standard.



Solution.—Let the counterweight be so adjusted that the centre of mass *C* is on the same side of the fixed point *O* as the disk *aa*. Let the entire mass be M, and the distance OC be l. Let OZ', OX', OY' be rectangular principal axes at O, fixed in the apparatus and rotating with it, and let the rectangular axes OZ vertical, and OX, OY horizontal, be fixed in space.



Let ω_x', ω_y', ω_z' be the angular velocity about the axes OX', OY', OZ' at any instant. Evidently ω_z' is always constant in magnitude, since the force Mg and the equal reaction at O always pass through OZ'.

Let θ be the angle of the axis OZ' of the disk with OZ at any instant and θ_1 the initial angle at the beginning of the motion. At this instant we have ω_x', ω_y' zero.

The moment of the weight Mg at C and the reaction at O about the axis OZ is always zero. Hence by the principle of conservation of moment of momentum (page 274) the moment of momentum about OZ is constant, and we have

$$I_z'\omega_z' \cos \theta_1 = I_z'\omega_z' \cos \theta + I_x'\omega_x' \cos ZOX' + I_y'\omega_y' \cos ZOY'.$$

In the present case of a rod and disk, every straight line through O at right angles to OZ' is a principal axis, and the moment of inertia about every such line is the same and equal to I'. We have then $I_x' = I_y' = I'$; and inserting the values of the cosines as given by equations (35), page 279, we have

$$I_z'\omega_z' (\cos \theta_1 - \cos \theta) = I'\omega_y' \sin \theta \sin \phi - I'\omega_x' \sin \theta \cos \phi. \quad . \quad . \quad (1)$$

The initial kinetic energy of rotation is $\frac{1}{2}I_z'\omega_z'^2$; and the initial potential energy with reference to a plane at a distance l below O, if C is on the same side of O as the disk, is $Mg(l + l \cos \theta_1)$; if C is on the opposite side of O from the disk, $Mg(l - l \cos \theta_1)$. The total initial energy is then

$$E_1 = \frac{1}{2}I_z'\omega_z'^2 + Mg(l \pm l \cos \theta_1).$$

The final kinetic energy of rotation (equation (11*a*), page 269) is

$$\frac{1}{2}I'\omega_x'^2 + \frac{1}{2}I'\omega_y'^2 + \frac{1}{2}I_z'\omega_z'^2;$$

and the final potential energy, if C is on the same side of O as the disk, is $Mg(l + l \cos \theta)$; if C is on the opposite side of O from the disk, $Mg(l - l \cos \theta)$. The total final energy is then

$$E_2 = \frac{1}{2}I'\omega_x'^2 + \frac{1}{2}I'\omega_y'^2 + \frac{1}{2}I_z'\omega_z'^2 + Mg(l \pm l \cos \theta).$$

If we disregard friction, we have by the principle of conservation of energy $E_1 = E_2$, or

$$\frac{1}{2}I'(\omega_x'^2 + \omega_y'^2) = \pm\, Mgl(\cos \theta_1 - \cos \theta), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the (+) sign is to be taken for centre of mass C on the same side of O as the disk, and the (−) sign when it is on the opposite side.

We have also from equations (33), page 279,

$$\left.\begin{aligned} \omega_x' &= \frac{d\theta}{dt} \sin \phi - \frac{d\psi}{dt} \sin \theta \cos \phi; \\ \omega_y' &= \frac{d\theta}{dt} \cos \phi + \frac{d\psi}{dt} \sin \theta \sin \phi; \\ \omega_z' &= \frac{d\psi}{dt} \cos \theta + \frac{d\phi}{dt}; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where (figure, page 278) $\frac{d\psi}{dt}$ is angular velocity of precession or rotation about OZ, and $\frac{d\theta}{dt}$ the angular velocity of nutation or rotation about the line of nodes ON.

Squaring and adding the first two of equations (3), we obtain

$$\omega_x'^2 + \omega_y'^2 = \frac{d\theta^2}{dt^2} + \frac{d\psi^2}{dt^2}\sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting (4) in (2) and the values of ω_x', ω_y' from (3) in (1), we have

$$I'\frac{d\theta^2}{dt^2} + I'\frac{d\psi^2}{dt^2}\sin^2\theta = \pm\, 2Mgl(\cos\theta_1 - \cos\theta); \quad . \quad . \quad . \quad . \quad (5)$$

$$I'\frac{d\psi}{dt}\sin^2\theta = I_z'\omega_z'(\cos\theta_1 - \cos\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also, from the last of equations (3),

$$\frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta = \omega_z'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Equations (5), (6) and (7) are the differential equations of motion of the gyroscope. When $\theta = \theta_1$, or at the beginning of motion, we have

$$\frac{d\psi}{dt} = 0, \quad \frac{d\theta}{dt} = 0 \quad \text{and} \quad \frac{d\phi}{dt} = \omega_z'.$$

From (6) we have

$$\frac{d\psi}{dt} = \frac{I_z'\omega_z'}{I'} \cdot \frac{\cos\theta_1 - \cos\theta}{\sin^2\theta}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and substituting in (5),

$$\frac{d\theta}{dt} = \sqrt{(\cos\theta_1 - \cos\theta)\left[\pm\frac{2Mgl}{I'} - \frac{I_z'^2\omega_z'^2}{I'^2\sin^2\theta}(\cos\theta_1 - \cos\theta)\right]}, \quad . \quad (9)$$

where the (+) sign is taken for C on the same side, and the (—) sign for C on the opposite side of O from the disk.

From (9) we see that for C on the same side of O as the disk $\frac{d\theta}{dt}$ is imaginary when θ is less than θ_1. Also for C on the opposite side of O from the disk, $\frac{d\theta}{dt}$ is imaginary when θ is greater than θ_1. The centre of mass then *always falls from its initial position* and can never rise above it.

From (8) then, if ω_z' is positive, that is, if the rotation of the disk looking from O to Z' is clockwise, $\frac{d\psi}{dt}$ is positive, or the rotation about OZ looking from O to Z is clockwise, if C is on the *same* side of O with the disk.

If C is on the opposite side of O from the disk, if ω_z' is positive $\frac{d\psi}{dt}$ is negative.

If C coincides with O, we have $l = 0$ and the angle θ_1 remains unchanged. Hence $\cos\theta - \cos\theta_1 = 0$ and $\frac{d\theta}{dt} = 0$, $\frac{d\psi}{dt} = 0$. The axis OZ' in such case remains stationary.

These conclusions can be illustrated by the apparatus (figure, page 285) by shifting the counterweight.

The length of the simple pendulum which would oscillate about OX' or

OY' in the same time as the gyroscope (if ω_z' were zero) is (page 179) $\frac{I'}{Ml}$. Let us call this length λ, so that

$$\lambda = \frac{I'}{Ml},$$

and let us put for convenience

$$\frac{I_z'^2\omega_z'^2}{2I'^2g} = \frac{2\beta^2}{\lambda}, \quad \text{or} \quad \beta^2 = \frac{I_z'^2\omega_z'^2}{4I'Mgl}.$$

Then equation (8) becomes

$$\sin^2\theta\frac{d\psi}{dt} = 2\beta\sqrt{\frac{g}{\lambda}}\,(\cos\theta_1 - \cos\theta), \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and equation (9) becomes

$$\sin^2\theta\frac{d\theta^2}{dt^2} = \frac{2g}{\lambda}[\pm\sin^2\theta - 2\beta^2(\cos\theta_1 - \cos\theta)]\,(\cos\theta_1 - \cos\theta). \quad . \quad . \quad (11)$$

If we put $\frac{d\theta}{dt} = 0$ we obtain the maximum and minimum values of θ. We have $\frac{d\theta}{dt} = 0$ when $\theta = \theta_1$, and this is the *minimum* value of θ, for we have just seen that θ cannot be less than θ_1 for C on same side of O as disk, nor greater than θ_1 for C on the opposite side. We shall also have $\frac{d\theta}{dt} = 0$ and θ a *maximum* when

$$\pm\sin^2\theta - 2\beta^2(\cos\theta_1 - \cos\theta) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

or denoting the maximum value of θ by θ_2,

$$\cos\theta_2 = \pm\beta^2 \mp \sqrt{1 \mp 2\beta^2\cos\theta_1 + \beta^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The upper signs are for C on same side of O as the disk, and the lower signs for C on the opposite side.

We see from (13) that the value of θ_2 depends upon β, and that θ_2 can be 0 or 180° or $\cos\theta_2 = +1$ or -1 only when $\beta = 0$. But β depends upon ω_z' and can be zero only when ω_z' is zero. Hence any velocity of rotation ω_z' of the disk, however minute, *is sufficient to prevent the axis from reaching the vertical OZ.* The self-sustaining power of the gyroscope is thus proved. From (12) we have, when θ is a maximum,

$$\cos\theta_1 - \cos\theta_2 = \frac{\pm\sin^2\theta_2}{2\beta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If β or ω_z' is very great, $\cos\theta_1 - \cos\theta_2$ is very small. Hence by increasing the value of ω_z' we see that $\theta_2 - \theta_1$ can be made less than any assignable quantity. This proves the apparently paradoxical result that the revolving disk does not *visibly fall.*

From (10) we see that for $\theta = \theta_1$, $\frac{d\psi}{dt}$ is zero. Hence the disk *must fall* in order to generate a rotation about OZ; but if ω_z' is great this fall is very minute, and *is not visible.*

The centre of mass, then, oscillates up and down between the minimum and maximum values of θ_1 and θ_2 given by (13), while the angular velocity $\frac{d\psi}{dt}$ of the centre of mass about OZ varies from $\frac{d\psi}{dt} = 0$, when the axis is in its initial position, to the maximum value given by (10), when we substitute the values of $\sin^2 \theta$ and $\cos \theta_1 - \cos \theta$ given by (12) and (14), viz.,

$$\frac{d\psi}{dt} = \frac{1}{2\beta}\sqrt{\frac{g}{\lambda}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The complete solution of the problem requires the integration of the differential equations (5), (6) and (7). This requires the use of elliptic functions. If, however, we *assume* that the velocity of rotation of the disk ω_z' is very great, and hence $\cos \theta_1 - \cos \theta_2$ or $\theta - \theta_1$ very minute, we may obtain integrals of (5) and (6) which will express the motion with all requisite accuracy.

Let us then assume ω_z' or β very large and $\theta - \theta_1$ very small, the centre of mass C being on the same side of O as the disk.

Let $\theta - \theta_1 = u$, or

$$\theta = \theta_1 + u, \qquad d\theta = du,$$

where u is a very small angle.

Then we have

$$\sin \theta = \sin \theta_1 \cos u + \cos \theta_1 \sin u;$$

$$\cos \theta = \cos \theta_1 \cos u - \sin \theta_1 \sin u.$$

Also by series, since u is a very small angle, neglecting higher powers of u than the square,

$$\sin u = u, \qquad \cos u = 1 - \frac{u^2}{2}.$$

Substituting, we have

$$\sin \theta = \sin \theta_1 \left(1 - \frac{u^2}{2}\right) + u \cos \theta_1;$$

$$\cos \theta = \cos \theta_1 \left(1 - \frac{u^2}{2}\right) - u \sin \theta_1.$$

Hence, neglecting higher powers of u than the square,

$$\sin^2 \theta = \sin^2 \theta_1 - u^2 \sin^2 \theta_1 + u^2 \cos^2 \theta_1 + 2u \sin \theta_1 \cos \theta_1;$$

$$\cos \theta_1 - \cos \theta = u \sin \theta_1 + \frac{1}{2}u^2 \cos \theta_1; \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and therefore

$$\frac{\cos \theta_1 - \cos \theta}{\sin^2 \theta} = \frac{u \sin \theta_1 + \frac{1}{2}u^2 \cos \theta_1}{\sin^2 \theta_1 + 2u \sin \theta_1 \cos \theta_1 + u^2 \cos^2 \theta_1 - u^2 \sin^2 \theta_1}; \quad . \quad (17)$$

$$\frac{(\cos \theta_1 - \cos \theta)^2}{\sin^2 \theta} = \frac{u^2 \sin^2 \theta_1}{\sin^2 \theta_1 + 2u \sin \theta_1 \cos \theta_1 + u^2 \cos^2 \theta_1 - u^2 \sin^2 \theta_1} = u^2. \quad (18)$$

Inserting (16) and (18) in (11), we obtain

$$\sqrt{\frac{g}{\lambda}} . \, dt = \frac{du}{\sqrt{2u \sin \theta_1 + u^2 (\cos \theta_1 - 4\beta^2)}}.$$

Since β or ω_z' has been assumed very great, $\cos \theta_1$ may be neglected in comparison with $4\beta^2$, and we have

$$\sqrt{\frac{g}{\lambda}} . \, dt = \frac{du}{\sqrt{2u \sin \theta_1 - 4\beta^2 u^2}} = \frac{1}{2\beta} \cdot \frac{du}{\sqrt{2u \frac{\sin \theta_1}{4\beta^2} - u^2}}. \quad . \quad . \quad (19)$$

Integrating, since when $t = 0$, $u = \theta - \theta_1 = 0$,

$$\sqrt{\frac{g}{\lambda}} \cdot t = \frac{1}{2\beta} \operatorname{versin}^{-1} \frac{4\beta^2}{\sin\theta_1} \cdot u = \frac{1}{2\beta} \cos^{-1}\left(1 - \frac{4\beta^2 u}{\sin\theta_1}\right). \quad . \quad . \quad (20)$$

Hence

$$u = \frac{\sin\theta_1}{4\beta^2}\left[1 - \cos\left(2\beta\sqrt{\frac{g}{\lambda}} \cdot t\right)\right]; \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

or since $\cos 2A = 1 - 2\sin^2 A$,

$$u = \frac{1}{2\beta^2} \sin\theta_1 \sin^2\left(\beta\sqrt{\frac{g}{\lambda}} \cdot t\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

We have from (17), neglecting the *square as well as higher powers of* u (which may be done without sensible error owing to the minuteness of u, though it could not be done in the foregoing values of dt and t, since β^2 is great when u is small),

$$\frac{\cos\theta_1 - \cos\theta}{\sin^2\theta} = \frac{u \sin\theta_1}{\sin^2\theta_1 + 2u \sin\theta_1 \cos\theta_1}.$$

The greatest possible value of $\sin\theta_1 \cos\theta_1$ is for $\theta_1 = 45°$, or

$$\sin\theta_1 \cos\theta_1 = \tfrac{1}{2}.$$

Since u is very small, we have then, neglecting $2u \sin\theta_1 \cos\theta_1$,

$$\frac{\cos\theta_1 - \cos\theta}{\sin^2\theta} = \frac{u}{\sin\theta_1};$$

and substituting in (10) we obtain

$$\frac{d\psi}{dt} = 2\beta\sqrt{\frac{g}{\lambda}} \cdot \frac{u}{\sin\theta_1}.$$

Inserting the value of u from (22), we have

$$\frac{d\psi}{dt} = \frac{1}{\beta}\sqrt{\frac{g}{\lambda}} \sin^2\left(\beta\sqrt{\frac{g}{\lambda}} \cdot t\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Integrating, since $d\psi = 0$ when $t = 0$, we obtain

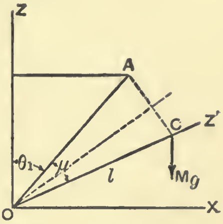
$$\psi = \frac{1}{2\beta}\sqrt{\frac{g}{\lambda}} \cdot t - \frac{1}{4\beta^2} \sin\left(2\beta\sqrt{\frac{g}{\lambda}} \cdot t\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Equations (22), (23), (24) give with all requisite accuracy the vertical angular depression of OZ', $u = \theta - \theta_1$, the horizontal angular velocity $\frac{d\psi}{dt}$, and the horizontal angle ψ at the end of any time t, provided ω_z' is very great.

Referring to (19), we see that *it is the differential equation of a cycloid generated by a circle whose angular diameter is* $\frac{\sin\theta_1}{2\beta^2}$. (Vol. I, *Kinematics*, page 157.)

When, starting from $t = 0$ (and therefore $u = 0$, $\frac{d\psi}{dt}$ and $\psi = 0$), u has its greatest value, we have from (20), (21), (23), (24)

$$t = \frac{\pi}{2\beta}\sqrt{\frac{\lambda}{g}}, \quad u = \frac{\sin\theta_1}{2\beta^2}, \quad \frac{d\psi}{dt} = \frac{1}{\beta}\sqrt{\frac{g}{\lambda}}, \quad \psi = \frac{\pi}{4\beta^2}.$$



After the expiration of the time $t = \frac{\pi}{\beta}\sqrt{\frac{\lambda}{g}}$, we have

$$u = 0, \quad \frac{d\psi}{dt} = 0, \quad \psi = \frac{\pi}{2\beta^2},$$

and OZ' has regained its original elevation and the horizontal velocity is zero.

The axis OZ' then moves as if it were the element of a right circular cone AOZ', the angle AOZ' being equal to u, which rolls on the cone ZOA, the angle ZOA being equal to θ_1.

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